Bargmann Transform With Application To Time-Dependent Schrödinger Equation

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Abstract: This article deals with the Bargmann transform as a new method to solve the time-dependent Schrödinger equation.

Index Terms: Bargmann transform, harmonic oscillator, integral transform, intertwinning operator, linear differential equation, quantum mechanics, Schrödinger equation.

1. INTRODUCTION

In 1926, Erwin Schrödinger ([1], [2], [3], and [4]) proposed a wave theory of quantum mechanics, along the lines of the de Broglie hypothesis. He considered the evolution of waves in time and showed that the energy levels of an atom can be considered as eigenvalues of a Hamiltonian operator. He also demonstrates that the wave model was equivalent to Heisenberg's matrix model. His formulation of the problem, later called the Schrödinger equation, takes into account both quantization and non-relativistic energy. The time-dependent Schrödinger equation is given by

$$\begin{cases} Hy(t,x) = \frac{\partial}{\partial t} y(t,x); & (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \\ y(0,x) = y_0(x); & y_0 \in L^2(\mathbb{R}). \end{cases} \qquad (1.1)$$

where H is the Hamiltonian operator of the system which correspond to the sum of the kinetic energies and the potential energies for all the particles in the system. In 1956, V. Bargmann [5] present an integral transform from the space of square integrable functions $L^2(\mathbb{R})$ to the Fock space [6]. This transform was called later Bargmann transform and a great number of research works has been done on it [7], [8], [9], [10], [11], [12], [13]. In this work, we use the Bargmann transform to solve the equation (1.1) where H is the Harmonic oscillator defined by $H = \frac{\partial^2}{\partial x^2} - a^2x^2$ which is one of the most famous model of Hamiltonian operator in the Quantum mechanics. The solution of this equation has been known for a long time as in [14] p.145 and is based essentially on the famous Mehler's formula [15], but our method is new. Indeed, we use the Bargmann transform as an intertwining operator that transports the harmonic oscillator to a complex Euler operator. This relationship is our main tool for solving the equation above.

This paper is outlined as follows

In section 2, some useful results about Bargmann transform are presented.

In section 3, we compute the exact solution of the Schrödinger equation (1.1).

In section 4, we compute explicitly the solution of heat Cauchy problems attached to the generalized real and complex Dirac operators by the method of section 3.

In section 5, we give some numerical results dealing with the solution of (1.1).

1 Bargmann transform

For $a > 0$, we define a Gaussian measure on $\mathbb{C}$ as follows

$$d\lambda_a(z) = \frac{a}{\pi} e^{-a|z|^2} dz.$$

The Fock space, denoted F_a^2, is the subspace of all entire functions in $L^2(\mathbb{C}, d\lambda_a)$ which is a Hilbert space with the inner product

$$< f, g > \int_{\mathbb{C}} f(z)\overline{g(z)} d\lambda_a(z). \qquad (2.2)$$

V. Bargmann, in [5], [16] has defined a mapping B, from the space of square integrable functions $L^2(\mathbb{R})$ to the Fock space F_1^2, called Bargmann transform, that is $\forall z \in \mathbb{C}, \forall f \in L^2(\mathbb{R})$

$$[Bf](z) = \left(\frac{1}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} f(x)\, e^{\sqrt{2}xz - \frac{1}{2}(x^2+z^2)}\, dx \qquad (2.3)$$

We use in this paper a parametrized form of the Bargmann transform given by K. Zhu [17] as follows

$$[B_a f](z) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} f(x)\, e^{2axz - ax^2 - \frac{a z^2}{2}}\, dx \qquad (2.4)$$

This mapping is an isometry from $L^2(\mathbb{R})$ to F_a^2, its inverse is given by

$$[B_a^{-1} f](x) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{\mathbb{C}} f(z)\, e^{2ax\bar{z} - ax^2 - \frac{a}{2}\bar{z}^2}\, d\lambda_a(z) \qquad (2.5)$$

Our aim in this section is to present some important results involving the Bargman transform and that will be useful to prove the principal theorems of this paper. These results are well known in the literature for the classical Bargmann transform B. We adapt here some of their proofs for the parametrized form B_a.

Lemma 2.1. ([17]) Let $f \in L^2(\mathbb{R})$ we have

1. $\left[B_{\frac{a}{2}}(xf)\right](z) = \left(\frac{1}{a}\frac{\partial}{\partial z} + \frac{z}{2}\right)\left[B_{\frac{a}{2}} f\right](z).$
2. $\left[B_{\frac{a}{2}}\left(\frac{\partial}{\partial x} f\right)\right](z) = \left(\frac{\partial}{\partial z} - \frac{a}{2}z\right)\left[B_{\frac{a}{2}} f\right](z).$
3. $\left[B_{\frac{a}{2}}\left(\left(\frac{\partial}{\partial x} - ax\right) f\right)\right](z) = -az\left[B_{\frac{a}{2}} f\right](z).$
4. $\left[B_{\frac{a}{2}}\left(\left(\frac{\partial}{\partial x} + ax\right) f\right)\right](z) = 2\frac{\partial}{\partial z}\left[B_{\frac{a}{2}} f\right](z).$

Let E_z^a and h_x^a be respectively the complex Euler operator and the quantum harmonic oscillator defined by

$$E_z^a = -2az\frac{\partial}{\partial z} - a, \quad h_x^a = \frac{\partial^2}{\partial x^2} - a^2 x^2 \qquad (2.6)$$

The following proposition is our principal tool in this paper, it is a direct consequence of lemma 2.1.

Proposition 2.1. We have

$$\left[B_{\frac{a}{2}}(h_x^a f)\right](z) = E_z^a \left[B_{\frac{a}{2}} f\right](z).$$

The following lemma gives the Bargmann transform of the

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Gaussian function.

Lemma 2.2. Let b, c, α and β be real numbers such that $b > 0$ and $|\alpha| < 1$, then the following lemma holds

1. $\left[B_{\frac{a}{2}}\left(e^{-bx^2+cx}\right)\right](z) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{2\pi}{a+2b}}\ e^{\frac{ac}{a+2b}z}\ e^{\frac{a}{4}\left(\frac{a-2b}{a+2b}\right)z^2}\ e^{\frac{c^2}{2(a+2b)}}$
2. $\left[B_{\frac{a}{2}}^{-1}\left(e^{\frac{a}{4}\alpha z^2+\beta z}\right)\right](z) = \left(\frac{\pi}{a}\right)^{\frac{1}{4}} \sqrt{\frac{a}{\pi(1+\alpha)}}\ e^{-\frac{a}{2}\left(\frac{1-\alpha}{1+\alpha}\right)x^2} e^{\left(\frac{2\beta}{1+\alpha}\right)x} e^{-\frac{\beta^2}{a(1+\alpha)}}$

Proof. For $c \in \mathbb{R}$ and $b > 0$ we can write

$$\left[B_{\frac{a}{2}}\left(e^{-bx^2+cx}\right)\right](z) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-bx^2+cx}\ e^{a\,xz-\frac{a}{2}x^2-\frac{a}{4}z^2}\ dx =$$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\left(b+\frac{a}{2}\right)\left(x-\frac{c}{a+2b}-\frac{a}{a+2b}z\right)^2+\left(b+\frac{a}{2}\right)\left(\frac{c^2}{(a+2b)^2}+\frac{a^2}{(a+2b)^2}z^2+\frac{2ac}{(a+2b)^2}z\right)-\frac{a}{4}z^2} dx$$

Then, $\left[B_{\frac{a}{2}}\left(e^{-bx^2+cx}\right)\right](z) =$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{4}} e^{\frac{c^2}{2(a+2b)}} e^{\left(\frac{a^2}{2(a+2b)}-\frac{a}{4}\right)z^2}\ e^{\frac{ac}{(a+2b)}z} \int_{-\infty}^{\infty} e^{-\left(b+\frac{a}{2}\right)\left(x-\frac{c}{a+2b}-\frac{a}{a+2b}z\right)^2} dx$$

From the formula (see lemma 2 of [8])

$$\int_{-\infty}^{\infty} e^{-\left(b+\frac{a}{2}\right)\left(x-\frac{c}{2b+a}-\frac{a}{2b+a}z\right)^2} dx = \sqrt{\frac{\pi}{b+\frac{a}{2}}}$$

we obtain

$$\left[B_{\frac{a}{2}}\left(e^{-bx^2+cx}\right)\right](z) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{2\pi}{a+2b}}\ e^{\frac{ac}{a+2b}z}\ e^{\frac{a}{4}\left(\frac{a-2b}{a+2b}\right)z^2}\ e^{\frac{c^2}{2(a+2b)}}.$$

Thus we obtain the first assertion of the lemma. The second assertion is a consequence of the first one if we put

$\alpha = \frac{a-2b}{a+2b}$ and $\beta = \frac{ac}{a+2b}$. □

Let $r \in \mathbb{R}$ such that $|r| < 1$ and k_r be the operator of $F_{\frac{a}{2}}^2$ defined by $[k_r g](z) = g(rz)$, for all $g \in F_{\frac{a}{2}}^2$ and $z \in \mathbb{C}$.

Theorem 2.1 Let $f \in L^2(\mathbb{R})$ and $r \in \mathbb{R}$ such that $|r| < 1$. We have

$$\left[B_{\frac{a}{2}}^{-1}\ k_r\ B_{\frac{a}{2}} f\right](x) = \frac{\sqrt{a}}{\sqrt{\pi}\sqrt{1-r^2}} \int_{\mathbb{R}} f(s) e^{-\frac{a}{2}\left(\frac{1+r^2}{1-r^2}\right)(x^2+s^2)} e^{\frac{2arsx}{1-r^2}} ds.$$

Proof. By definition, $\left[B_{\frac{a}{2}}^{-1} k_r B_{\frac{a}{2}} f\right](x) =$

$$\sqrt{\frac{a}{\pi}} \int_{\mathbb{C}} e^{ax\bar{z}-\frac{a}{2}x^2-\frac{a}{4}\bar{z}^2} \int_{\mathbb{R}} f(s)\ e^{asrz-\frac{a}{2}s^2-\frac{a}{4}r^2z^2}\ ds\ d\lambda_{\frac{a}{2}}(z).$$

Using the change of variables $s' = rs$, we get

$\left[B_{\frac{a}{2}}^{-1} k_r B_{\frac{a}{2}} f\right](x) =$

$$\sqrt{\frac{a}{\pi}}\ \frac{1}{r} \int_{\mathbb{C}} e^{ax\bar{z}-\frac{a}{2}x^2-\frac{a}{4}\bar{z}^2} \int_{\mathbb{R}} f\left(\frac{s'}{r}\right) e^{as'z-\frac{as'^2}{2r^2}-\frac{a}{4}r^2 z^2}\ ds'\ d\lambda_{\frac{a}{2}}(z)$$

$$= \sqrt{\frac{a}{\pi}}\ \frac{1}{r} \int_{\mathbb{R}} f\left(\frac{s'}{r}\right)\ e^{-\frac{as'^2}{2r^2}} \int_{\mathbb{C}} e^{as'z-\frac{a}{4}r^2 z^2+ax\bar{z}-\frac{a}{2}x^2-\frac{a}{4}\bar{z}^2}\ d\lambda_{\frac{a}{2}}(z)\ ds'.$$

We can easily check that applying Fubini's theorem in the previous line is justified. Now, we notice that

$$\left[B_{\frac{a}{2}}^{-1} k_r B_{\frac{a}{2}} f\right](x) = \left(\frac{a}{\pi}\right)^{\frac{1}{4}} \frac{1}{r} \int_{\mathbb{R}} f\left(\frac{s'}{r}\right) e^{-\frac{as'^2}{2r^2}} \left[B_{\frac{a}{2}}^{-1}\left(e^{as'z-\frac{a}{4}r^2 z^2}\right)\right](x) ds'.$$

We deduce from lemma 2.2 that

$$\left[B_{\frac{a}{2}}^{-1}\left(e^{as'z-\frac{a}{4}r^2 z^2}\right)\right](x) = \left(\frac{\pi}{a}\right)^{\frac{1}{4}} \frac{\sqrt{a}}{\sqrt{\pi}\sqrt{1-r^2}}\ e^{-\frac{a}{2}\left(\frac{1+r^2}{1-r^2}\right)x^2} e^{\frac{2as'}{1-r^2}x}\ e^{-\frac{as'^2}{1-r^2}}.$$

So, we obtain

$\left[B_{\frac{a}{2}}^{-1} k_r B_{\frac{a}{2}} f\right](x) =$

$$\frac{1}{r} \frac{\sqrt{a}}{\sqrt{\pi}\sqrt{1-r^2}} \int_{\mathbb{R}} f\left(\frac{s'}{r}\right)\ e^{-\frac{as'^2}{2r^2}}\ e^{-\frac{a}{2}\left(\frac{1+r^2}{1-r^2}\right)x^2}\ e^{\frac{2as'}{1-r^2}x}\ e^{-\frac{as'^2}{1-r^2}} ds'.$$

Set again $s' = rs$, we obtain

$\left[B_{\frac{a}{2}}^{-1} k_r B_{\frac{a}{2}} f\right](x)$

$$= \frac{\sqrt{a}}{\sqrt{\pi}\sqrt{1-r^2}} \int_{\mathbb{R}} f(s)\ e^{-\frac{a}{2}s^2} e^{-\frac{a}{2}\left(\frac{1+r^2}{1-r^2}\right)x^2} e^{\frac{2ar\,s}{1-r^2}x}\ e^{-\frac{ar^2 s^2}{1-r^2}}\ ds.$$

Hence

$$\left[B_{\frac{a}{2}}^{-1} k_r B_{\frac{a}{2}} f\right](x) = \frac{\sqrt{a}}{\sqrt{\pi}\sqrt{1-r^2}} \int_{\mathbb{R}} f(s)\ e^{-\frac{a}{2}\left(\frac{1+r^2}{1-r^2}\right)(x^2+s^2)}\ e^{\frac{2ar\,s}{1-r^2}x}\ ds. \ \square$$

2 TIME-DEPENDENT SCHRÖDINGER EQUATION

Our purpose in this section is to give explicit solution of the time-dependent Schrödinger equation (1.1). For this aim, we consider this partial differential equation

$$\begin{cases} E_z^a Y(t,z) = \frac{\partial}{\partial t} Y(t,z); & (t,z) \in \mathbb{R}_+^* \times \mathbb{C} \\ Y(0,z) = Y_0(z); & Y_0 \in F_{\frac{a}{2}}^2, \end{cases} \quad (3.1)$$

where E_z^a is the complex Euler operator defined in (2.6).

Lemma 3.1 The solution of the Cauchy problem (3.1) associated to the complex Euler operator is given by the formula

$$Y(t,z) = e^{-at}\ Y_0(e^{-2at}z).$$

Proof. The fact that this function satisfies (3.1) can be checked directly. □

Theorem 3.1 The time-dependent Schrödinger equation (1.1) has the unique solution given by

$$y(t,x) = \int_{\mathbb{R}} K_a(x,s,t)\ y_0(s)\ ds$$

where

$$K_a(x,s,t) = \sqrt{\frac{a}{2\pi}} \frac{1}{\sqrt{\sinh(2at)}}\ e^{\left(-\frac{a}{2}(x^2+s^2)\coth(2at)+\frac{axs}{\sinh(2at)}\right)} \quad (3.3)$$

Proof. Let y be a solution of (1.1). Then, by applying the Bargmann transform $B_{\frac{a}{2}}$, we obtain

$$\begin{cases} \left[B_{\frac{a}{2}}\left(h_x^a\ y(t,x)\right)\right](z) = \frac{\partial}{\partial t}\left[B_{\frac{a}{2}}\left(y(t,x)\right)\right](z); & (t,z) \in \mathbb{R}_+^* \times \mathbb{C} \\ \left[B_{\frac{a}{2}}\left(y(0,x)\right)\right](z) = \left[B_{\frac{a}{2}} y_0\right](z); & y_0 \in L^2(\mathbb{R}). \end{cases}$$

Proposition 2.1 assures that

$$\begin{cases} \left(-2az\frac{\partial}{\partial z} - a\right)\left[B_{\frac{a}{2}}\left(y(t,x)\right)\right](z) = \frac{\partial}{\partial t}\left[B_{\frac{a}{2}}\left(y(t,x)\right)\right](z); \\ \left[B_{\frac{a}{2}}\left(y(0,x)\right)\right](z) = \left[B_{\frac{a}{2}}(y_0)\right](z); \quad y_0 \in L^2(\mathbb{R}). \end{cases}$$

Now, lemma 3.1 gives

$$\left[B_{\frac{a}{2}} y\right](t,z) = e^{-at}\left[B_{\frac{a}{2}} y_0\right](e^{-2at}z), \qquad (t,z) \in \mathbb{R}_+^* \times \mathbb{C}.$$

Thus,

$$y(t,x) = \left[B_{\frac{a}{2}}^{-1}\left(e^{-at}\left[B_{\frac{a}{2}} y_0\right](e^{-2at}z)\right)\right](x)$$

Theorem 2.1 now implies that

$$y(t,x) = \sqrt{\frac{a}{\pi}} \frac{e^{-at}}{\sqrt{1-e^{-4at}}} \int_{\mathbb{R}} y_0(s) e^{-\frac{a}{2}\left(\frac{1+e^{-4at}}{1-e^{-4at}}\right)(x^2+s^2)}\ e^{\frac{2ae^{-2at}}{1-e^{-4at}}sx} ds$$

$$= \sqrt{\frac{a}{2\pi}} \frac{1}{\sqrt{\sinh(2at)}} \int_{\mathbb{R}} y_0(s)\ e^{-\frac{a}{2}(x^2+s^2)\coth(2at)}\ e^{\frac{axs}{\sinh(2at)}}\ ds. \ \square$$

3 APPLICATIONS

In this section, we make use of the results of lemma 2.1 to compute explicitly the following Cauchy problems associated to the generalized complex and real Dirac operators

$$\begin{cases} D_z^a U(t,z) = \frac{\partial}{\partial t} U(t,z); & (t,z) \in \mathbb{R}_+^* \times \mathbb{C}, \\ U(0,z) = U_0(z); & U_0 \in F_{\frac{a}{2}}^2 \end{cases} \quad (4.1)$$

and

$$\begin{cases} d_x^a u(t,x) = \frac{\partial}{\partial t} u(t,x); & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ u(0,x) = u_0(x); & u_0 \in L^2(\mathbb{R}). \end{cases} \quad (4.2)$$

where the generalized complex and real Dirac operators are given by

$$D_z^a = \frac{1}{a}\frac{\partial}{\partial z} + \frac{z}{2}, \quad \text{and} \quad d_x^a = \frac{\partial}{\partial x} - ax. \quad (4.3)$$

Theorem 4.1 The solution of the Cauchy problem (4.1) associated to the generalized complex Dirac operator is given by the formula

$$U(t,z) = e^{\frac{zt}{2}} e^{\frac{t^2}{4a}} U_0\left(z + \frac{t}{a}\right).$$

Proof Let U be a solution of (4.1). Then, by applying the inverse of the Bargmann transform $B_{\frac{a}{2}}^{-1}$, we obtain

$$\begin{cases} B_{\frac{a}{2}}^{-1}\left(\left(\frac{1}{a}\frac{\partial}{\partial z} + \frac{z}{2}\right) U(t,z)\right) = \frac{\partial}{\partial t} B_{\frac{a}{2}}^{-1}\left(U(t,z)\right); \ (t,z) \in \mathbb{R}_+^* \times \mathbb{C}, \\ B_{\frac{a}{2}}^{-1}(U(0,z)) = B_{\frac{a}{2}}^{-1} U_0; \qquad U_0 \in F_{\frac{a}{2}}^2. \end{cases}$$

Using lemma 2.1 we get

$$\begin{cases} x\left[B_{\frac{a}{2}}^{-1} U\right](t,x) = \frac{\partial}{\partial t}\left[B_{\frac{a}{2}}^{-1} U\right](t,x); \ (t,x) \in \mathbb{R}_+^* \times \mathbb{R} \\ \left[B_{\frac{a}{2}}^{-1} U\right](0,x) = \left[B_{\frac{a}{2}}^{-1} U_0\right](x); \quad U_0 \in F_{\frac{a}{2}}^2. \end{cases}$$

Thus $B_{\frac{a}{2}}^{-1} U$ satisfies the formula

$$\left[B_{\frac{a}{2}}^{-1} U\right](t,x) = e^{tx}\left[B_{\frac{a}{2}}^{-1} U_0\right](x).$$

This implies that

$$U(t,z) = \left[B_{\frac{a}{2}}\left(e^{tx} B_{\frac{a}{2}}^{-1} U_0\right)\right](z) =$$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} e^{axz - \frac{a}{2}x^2 - \frac{a z^2}{4}} e^{tx} \int_{\mathbb{C}} U_0(v) e^{ax\bar{v} - \frac{a}{2}x^2 - \frac{a}{4}\bar{v}^2} \, d\lambda_{\frac{a}{2}}(v) dx =$$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} \int_{\mathbb{C}} e^{tx} U_0(v) e^{axz - \frac{a}{2}x^2 - \frac{a z^2}{4}} e^{ax\bar{v} - \frac{a}{2}x^2 - \frac{a}{4}\bar{v}^2} \, d\lambda_{\frac{a}{2}}(v) dx =$$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}} e^{-\frac{a z^2}{4}} e^{\frac{a\,(z + \frac{t}{a})^2}{4}} \int_{\mathbb{R}} \int_{\mathbb{C}} U_0(v) e^{ax\left(z + \frac{t}{a}\right) - \frac{a}{2}x^2} e^{-\frac{a\,(z + \frac{t}{a})^2}{4}} e^{ax\bar{v} - \frac{a}{2}x^2 - \frac{a}{4}\bar{v}^2} \, d\lambda_{\frac{a}{2}}(v) dx$$

Consequently,

$$U(t,z) = e^{\frac{zt}{2}} e^{\frac{t^2}{4a}}\left[B_{\frac{a}{2}}\left(B_{\frac{a}{2}}^{-1} U_0\right)\right]\left(z + \frac{t}{a}\right) = e^{\frac{zt}{2}} e^{\frac{t^2}{4a}} U_0\left(z + \frac{t}{a}\right). \square$$

Theorem 4.2 The solution of the Cauchy problem (4.2) associated to the generalized real Dirac operator is given by the formula

$$u(t,x) = e^{-axt} e^{\frac{-at^2}{2}} u_0(x+t).$$

Proof Let u be a solution of (4.2). Then, by applying the Bargmann transform $B_{\frac{a}{2}}$, we obtain

$$\begin{cases} B_{\frac{a}{2}}\left(\left(\frac{\partial}{\partial x} - ax\right) u(t,x)\right) = \frac{\partial}{\partial t} B_{\frac{a}{2}}(u(t,x)); & (t,x) \in \mathbb{R}_+^* \times \mathbb{R}, \\ B_{\frac{a}{2}}(u(0,x)) = B_{\frac{a}{2}}(u_0); & u_0 \in L^2(\mathbb{R}). \end{cases}$$

Using lemma 2.1 we obtain

$$\begin{cases} -az\left[B_{\frac{a}{2}} u\right](t,z) = \frac{\partial}{\partial t}\left[B_{\frac{a}{2}} u\right](t,z); & (t,z) \in \mathbb{R}_+^* \times \mathbb{C} \\ \left[B_{\frac{a}{2}} u\right](0,z) = \left[B_{\frac{a}{2}} u_0\right](z); & u_0 \in L^2(\mathbb{R}). \end{cases}$$

Then $B_{\frac{a}{2}} u$ satisfies the formula

$$\left[B_{\frac{a}{2}} u\right](t,z) = e^{-azt}\left[B_{\frac{a}{2}} u_0\right](z).$$

This implies that

$$u(t,x) = \left[B_{\frac{a}{2}}^{-1}\left(e^{-azt} B_{\frac{a}{2}} u_0\right)\right](x)$$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} \int_{\mathbb{R}} e^{-azt} u_0(s) e^{asz - \frac{a}{2}s^2 - \frac{a z^2}{4}} e^{ax\bar{z} - \frac{a}{2}x^2 - \frac{a}{4}\bar{z}^2} ds \, d\lambda_a(z)$$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} \int_{\mathbb{R}} u_0(s) \, e^{az(s-t) - \frac{a}{2}s^2 - \frac{a z^2}{4}} e^{ax\bar{z} - \frac{a}{2}x^2 - \frac{a}{4}\bar{z}^2} ds \, d\lambda_a(z)$$

By setting $s' = s - t$, we obtain $u(t,x) =$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} \int_{\mathbb{R}} u_0(s'+t) e^{azs'} e^{-\frac{a}{2}s'^2} e^{-\frac{a}{2}t^2} e^{-as't} e^{-\frac{a z^2}{4}} e^{ax\bar{z} - \frac{a}{2}x^2 - \frac{a}{4}\bar{z}^2} ds' d\lambda_a(z)$$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{4}} B_{\frac{a}{2}}^{-1}\left(\int_{\mathbb{R}} u_0(s'+t) \, e^{-\frac{a}{2}t^2} e^{-as't} e^{azs'} e^{-\frac{a}{2}s'^2} e^{-\frac{a z^2}{4}} ds'\right) \quad (4.4)$$

$$= \left[B_{\frac{a}{2}}^{-1}\left(B_{\frac{a}{2}}\left(u_0(s'+t) e^{-\frac{a}{2}t^2} e^{-a s' t}\right)\right)\right](x)$$

$$= u_0(x+t) \, e^{-\frac{a}{2}t^2} e^{-a x t}. \square$$

4 NUMERICAL RESULT

We compute here the numerical values of the propagator of the time-dependent Schrödinger equation $K_a(x,y,t)$ for $(a = 1, t = 1, x = 1..5$ and $y = 1..5)$ (figure1). Secondly, we give the graphical representation of $K_a(x,y,t)$ for $a = 1, t = 1$, (figure 2). And finally, we make a comparison between the graphical representation of $K_a(x,0,t)$ for $a = 1, t = 1$ and the free Schrödinger propagator (normal distribution) (figure 3).

Figure 1: *Value table of the propagator* $K_a(x,y,1)$

x \ y	1	2	3	4	5
1	0.138326	0.0384501	0.00378784	0.000132247	1.63636×10^{-6}
2	0.0384501	0.0140811	0.00182756	0.0000840641	1.3704×10^{-6}
3	0.00378784	0.00182756	0.000312503	0.0000189381	4.06741×10^{-7}
4	0.000132247	0.0000840641	0.0000189381	1.51203×10^{-6}	4.27845×10^{-8}
5	1.63636×10^{6}	1.3704×10^{6}	4.06741×10^{6}	4.27845×10^{6}	1.59498×10^{-9}

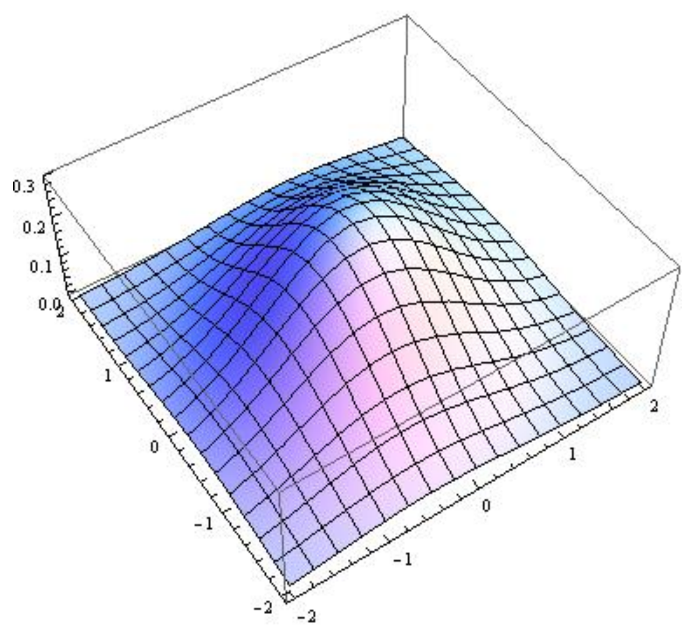

Figure 2: *three dimension graph*

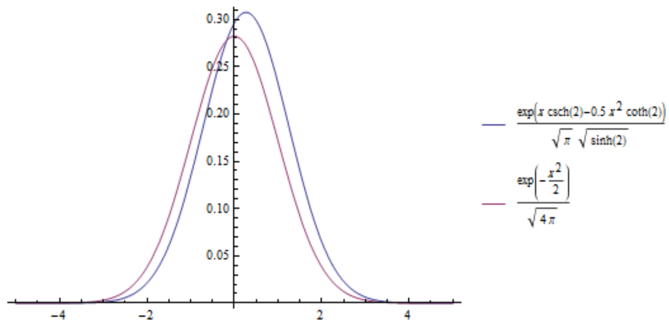

Figure 3: *graphic comparison with normal disribution*

5 CONCLUSION

In this work, we present a new method, based on the famous Bargmann transform, to solve the time-dependent Schrödinger equation. This method might be useful for solving a large class of partial differential equations, especially those concerning the operators that can be intertwinned by the Bargmann transform.

ACKNOWLEDGMENT

The author(s) received no financial support for the research, authorship and publication of this article.

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