CONNECTED DOMINATION DECOMPOSITION OF WEB AND SUNLET GRAPHS

E. Ebin Raja Merly, D. Jeya Jothi

Abstract: Let $G = (V, E)$ be a simple connected graph with p vertices and q edges. A Decomposition (G_1, G_2 ,...,G_n) of G is said to be connected domination decomposition (CDD) if (i) $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$ (ii) Each G_i is connected . (iii) $\gamma_c(G_i) = i$, $1 \le i \le n$. In this paper, we establish connected domination decomposition of web and sunlet graphs. In particular, we establish Connected domination Decomposition of degree splitting graph of sunlet graph and line graph of sunlet graph.

Keywords: Connected Domination, Decomposition, Connected Domination Decomposition.

1. INTRODUCTION

All basic terminologies from graph theory are used in this paper in the sense of Frank Harary, S.T.Hedetniemi. By a graph considered here are simple undirected graph without loops or multiple edges. As usual p and q denote the vertices and edges of a graph G respectively.

A set $D \subseteq V$ of vertices in a graph G is a **dominating set** if every vertex v in $V - D$ is adjacent to a vertex in D. The minimum cardinality of a dominating set of G is called the **domination number** of G and is denoted by $\gamma(G)$.

A Dominating set D of a graph G is a **connected dominating set** if the induced subgraph $< D >$ is connected. The number of vertices in a minimum connected dominating set is defined as the **Connected Domination number** of a graph G and it is denoted by $\gamma_c(G)$. Let $G = (V, E)$ be a simple connected graph with p vertices and q edges. If $G_1, G_2, ..., G_n$ are connected edge disjoint subgraphs of G with $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$, then ($G_1$, G_2, ..., G_n) is said to be a **Decomposition** of G.

[1]A **web graph** as a stacked prism graph $Y_{n+1,3}$ with the edges of the outer cycle removed. Let G be a graph with $V = T_1 \cup T_2 \cup \cdots \cup T_t \cup T$, where each T_i is a set of vertices having at least two vertices and having the same degree and $T = V - \cup T_i$. [2]The **Line graph L(G) of a graph G** is that graph whose vertices can be put in one-to one correspondence with the edges of G in such a way that two vertices of L(G) are adjacent if and only if the corresponding edges of G are adjacent. [7]The ***Degree splitting graph of*** G denoted by DS(G) is obtained from G by adding vertices $w_1, w_2, . . . , w_t$ and joining w_i to each vertex of T_i ($1 \le i \le t$). [8]The ***t - sunlet graph*** is the graph on $2t$ vertices obtained by attaching t pendant edges to a cycle graph C_t and is denoted by SL_t.

2. CONNECTED DOMINATION DECOMPOSITION OF WEB GRAPH

Definition 2.1[3]: A Decomposition (G_1, G_2, ..., G_n) of G is said to be **Connected Domination Decomposition(CDD)** if (*i*) $E(G) = E(G_1) \cup E(G_2) \cup ... \cup E(G_n)$ (*ii*) Each G_i is connected (*iii*) $\gamma_c(G_i) = i$, $1 \le i \le n$.

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Theorem 2.2*:* The web graph $WB_{t(4t-1)}$ admits CDD(G_1, G_2,..., G_{4t-1}).

Proof: Let $V(WB_{t(4t-1)}) = \{ v_1, v_2, ..., v_{t(4t-1)}, v_1', v_2', ..., v_{t(4t-1)}', v_1'', v_2'', ..., v_{t(4t-1)}''\}$ such that each v_i's are degree 4, each v_i' 's are degree 3 and each v_i'' 's are pendant vertices. Take $R_1 = v_1 v_2 ... v_{t(4t-1)} v_1$ and $R_2 = v_1' v_2' ... v_{t(4t-1)}' v_1'$. Consider $e_i = v_i v_i'$, $e_i' = v_i v_i''$, $i = 1, 2, 3, ..., t(4t-1)$.

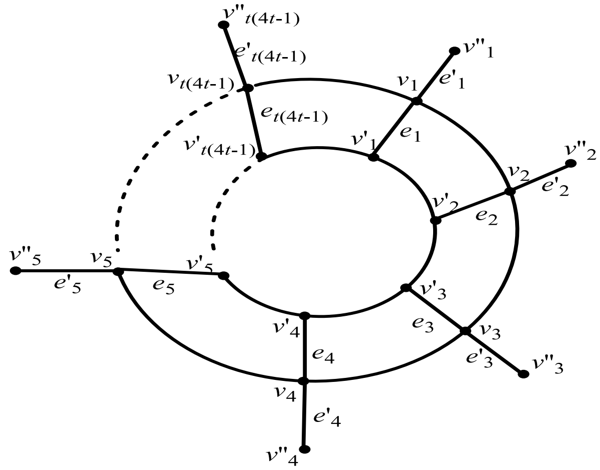

Figure 2.1: Web graph $WB_{t(4t-1)}$

Now, $WB_{t(4t-1)} = R_1' \cup R_2'$, where $R_1' = R_1 \cup \{ e_i' / i = 1, 2, 3, ..., t(4t-1)\}$ and $R_2' = R_2 \cup \{e_i / i = 1, 2, 3, ..., t(4t-1)\}$.

Case (i): $t = 1$

$R_1' = G_1 \cup G_2$, where G_1= Shortest $v_3 - v_1$ path $\cup \{e_1'\}$, G_2 = Longest $v_1 - v_3$ path $\cup \{e_2', e_3'\}$. Then $\gamma_c(G_1) = 1$, $\gamma_c(G_2) = 2$. Thus R_1' can be decomposed into (G_1, G_2) and satisfies the condition $\gamma_c(G_i) = i$, $i = 1, 2$. Also, $R_2' = G_3$, where $G_3 = \{v_1' v_2'\} \cup \{v_2' v_3'\} \cup \{v_3' v_1'\} \cup \{e_1, e_2, e_3\}$. Then $\gamma_c(G_3) = 3$. Thus R_2' can be decomposed into (G_3) and satisfies the condition $\gamma_c(G_3) = 3$. Hence WB_3 can be decomposed into (G_1, G_2, G_3) and $\gamma_c(G_i) = i$, $i = 1, 2, 3$

Case (ii): $t > 1$

Let $R_1' - (G_1 \cup G_2) = R_1''$ (say), where G_1 = Shortest $v_{t(4t-1)} - v_1$ path $\cup \{e_1'\}$ and G_2 = Shortest $v_1 - v_3$ path $\cup \{e_2', e_3'\}$. Then $\gamma_c(G_1) = 1$ and $\gamma_c(G_2) = 2$.

Now, $R_1'' = (\cup_{k=2}^{t} G_{4k-4}) \cup (\cup_{k=2}^{t} G_{4k-1})$, where

G_{4k-4}= $v_{4k^2-9k+5} - v_{4k^2-5k+1}$ path ∪ {e'_{4k^2-9k+6}, e'_{4k^2-9k+7}, …, e'_{4k^2-5k+1}} and G_{4k-1} = $v_{4k^2-5k+1} - v_{4k^2-k}$ path ∪ {e'_{4k^2-5k+2}, e'_{4k^2-5k+3}, …, e'_{4k^2-k}}.
Then $\gamma_c(G_{4k-4}) = 4k - 4$ and $\gamma_c(G_{4k-1}) = 4k - 1$. Therefore, R''_1 can be decomposed into (G_4, G_8, …, G_{4t-4}) and (G_7, G_{11}, ..., G_{4t-1}). Thus R'_1 can be decomposed into (G_1, G_2), (G_4, G_8,…, G_{4t-4}) and (G_7, G_{11}, ..., G_{4t-1}).

Let $R''_2 = R'_2 - G_3$, where G_3 = Shortest $v'_{4t^2-t} - v'_3$ path ∪ {e_1, e_2, e_3}. Then $\gamma_c(G_3) = 3$. Therefore, $R''_2 = (\cup_{k=2}^{t} G_{4k-3}) \cup (\cup_{k=2}^{t} G_{4k-2})$, where

$G_{4k-3} = v_{4k^2-9k+5} - v_{4k^2-5k+2}$ path ∪ {e_{4k^2-9k+6}, e_{4k^2-9k+7}, …, e_{4k^2-5k+2}} and

$G_{4k-2} = v_{4k^2-5k+2} - v_{4k^2-k}$ path ∪ {e_{4k^2-5k+3}, e_{4k^2-5k+4}, …, e_{4k^2-k}}.

Then $\gamma_c(G_{4k-3}) = 4k - 3$ and $\gamma_c(G_{4k-2}) = 4k - 2$. Therefore, R''_2 can be decomposed into (G_5, G_9, ..., G_{4t-3}), (G_6, G_{10}, ..., G_{4t-2}). Thus R'_2 can be decomposed into (G_3), (G_5, G_9, ..., G_{4t-3}), (G_6, G_{10}, ..., G_{4t-2}). Hence web graph $WB_{t(4t-1)}$ admits CDD(G_1, G_2, …, G_{4t-1}).

Remark 2.3: The web graph $WB_{t(4t+1)}$ admits CDD(G_1, G_2, …, G_{4t}).

Theorem 2.4: If Web graph $WB_{t(4t-1)}$ admits CDD(G_1, G_2, …, G_{4t-1}), then the following conditions are hold.

(i) $\gamma_c(WB_{t(4t-1)}) = \frac{1}{2}\sum_{i=1}^{4t-1}\gamma_c(G_i)$.

(ii) $2\sum_{i=1}^{4t-1}\gamma_c(G_i) = q(WB_{t(4t-1)})$

Proof: (i). Let V = {v_1, v_2, …, $v_{t(4t-1)}$} be a vertex set such that each v_i's are degree 4. Then V is a minimum connected dominating set of $WB_{t(4t-1)}$. Therefore, $\gamma_c(WB_{t(4t-1)}) = t(4t - 1)$
Now,

$\frac{1}{2}\sum_{i=1}^{4t-1}\gamma_c(G_i) = \frac{1}{2}[\gamma_c(G_1) + \gamma_c(G_2) + \dots + \gamma_c(G_{4t-1})]$
$= \frac{1}{2}[1 + 2 + 3 + \dots + (4t - 1)] = t(4t - 1)$
$\frac{1}{2}\sum_{i=1}^{4t-1}\gamma_c(G_i) = \gamma_c(WB_{t(4t-1)})$

(ii). Since $\gamma_c(G_i) = i$, i = 1, 2, 3, …, $4t - 1$, i vertices form a minimum connected dominating set of G_i. Each G_i, i = 1, 2, 3, …, $4t - 1$ consist of $i + 1$ pendant edges. Therefore,
$q(G_i) = i + 1 + i - 1 = 2\gamma_c(G_i)$, i = 1, 2, 3, …, $4t - 1$.
Now,
$2\sum_{i=1}^{4t-1}\gamma_c(G_i) = 2(\gamma_c(G_1) + \gamma_c(G_2) + \dots + \gamma_c(G_{4t-1}))$
$= q(WB_{t(4t-1)})$

Remark 2.5: If $WB_{t(4t+1)}$ admits CDD(G_1, G_2, …, G_{4t}), then the following conditions are hold.

(i) $\gamma_c(WB_{t(4t+1)}) = \frac{1}{2}\sum_{i=1}^{4t}\gamma_c(G_i)$.

(ii) $2\sum_{i=1}^{4t}\gamma_c(G_i) = q(WB_{t(4t+1)})$

3. CONNECTED DOMINATION DECOMPOSITION OF SUNLET GRAPH

Theorem 3.1: The Sunlet graph $SL_{\frac{(t+1)(t+2)}{2}}$ admits CDD(G_1, G_2, …, G_{t+1}).

Proof: Let v_1, v_2, …, $v_{\frac{(t+1)(t+2)}{2}}$ be the vertices of degree 3 and v'_1, v'_2, …, $v'_{\frac{(t+1)(t+2)}{2}}$ be the pendant vertices of $SL_{\frac{(t+1)(t+2)}{2}}$. Consider $e_1 = v_1v'_1$, $e_2 = v_2v'_2$, ..., $e_{\frac{(t+1)(t+2)}{2}} = v_{\frac{(t+1)(t+2)}{2}}v'_{\frac{(t+1)(t+2)}{2}}$.

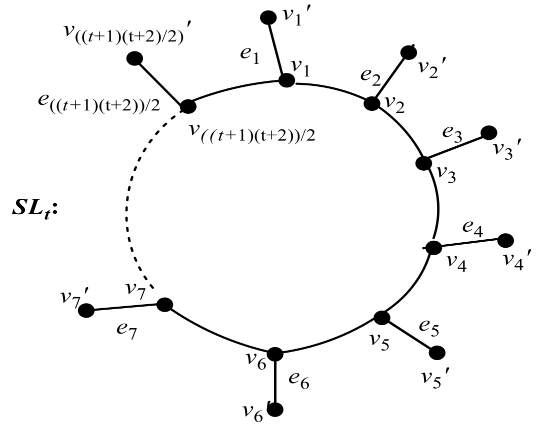

Figure 3.1: Sunlet graph $SL_{\frac{(t+1)(t+2)}{2}}$

Let $R_1 = SL_{\frac{(t+1)(t+2)}{2}} - G_1$, where E($G_1$) = {$v_{\frac{(t+1)(t+2)}{2}}v_1$, $v_1v'_1$}. Then $\gamma_c(G_1) = 1$. Now, $R_1 = \cup_{k=1}^{t} G_{k+1}$, where $G_{k+1} = v_{\frac{k(k+1)}{2}} - v_{\frac{k(k+3)}{2}+1}$ path ∪ {$e_{\frac{k(k+1)}{2}+1}$, $e_{\frac{k(k+1)}{2}+2}$, …, $e_{\frac{k(k+3)}{2}+1}$}. Then $\gamma_c(G_{k+1}) = k + 1$. Thus R_1 can be decomposed into (G_2, G_3, …, G_{t+1}) and satisfies the condition $\gamma_c(G_i) = i$, i = 2, 3, …, t + 1. Hence Sunlet graph $SL_{\frac{(t+1)(t+2)}{2}}$ admits CDD(G_1, G_2, …, G_{t+1}).

Theorem 3.2: If $SL_{\frac{(t+1)(t+2)}{2}}$ admits CDD(G_1, G_2, …, G_{t+1}), then the following conditions are hold.

(i) $\gamma_c(SL_{\frac{(t+1)(t+2)}{2}}) = \sum_{i=1}^{t+1}\gamma_c(G_i)$.

(ii) $2\sum_{i=1}^{t+1}\gamma_c(G_i) = q(SL_{\frac{(t+1)(t+2)}{2}})$

Proof: (i). Let v_1, v_2, …, $v_{\frac{(t+1)(t+2)}{2}}$ be the vertices of degree 3 in $SL_{\frac{(t+1)(t+2)}{2}}$. Then v_1, v_2, …, $v_{\frac{(t+1)(t+2)}{2}}$ is a minimum connected dominating set of $SL_{\frac{(t+1)(t+2)}{2}}$. Therefore, $\gamma_c(SL_{\frac{(t+1)(t+2)}{2}}) = \frac{(t+1)(t+2)}{2}$
Now,
$\sum_{i=1}^{t+1}\gamma_c(G_i) = \gamma_c(G_1) + \gamma_c(G_2) + \dots + \gamma_c(G_{t+1})$
$= 1 + 2 + 3 + \dots + (t + 1) = \gamma_c(SL_{\frac{(t+1)(t+2)}{2}})$
i.e, $\gamma_c(SL_{\frac{(t+1)(t+2)}{2}}) = \sum_{i=1}^{t+1}\gamma_c(G_i)$.

(ii). Since $\gamma_c(G_i) = i$, i = 1, 2, 3, …, $t + 1$, i vertices form a minimum connected dominating set of G_i. Each G_i, i = 1, 2, 3, …, $t + 1$ consist of $i + 1$ pendant edges. Therefore,
$q(G_i) = i + 1 + i - 1 = 2\gamma_c(G_i)$, i = 1, 2, 3, …, $t + 1$
Also,
$2\sum_{i=1}^{t+1}\gamma_c(G_i) = 2(\gamma_c(G_1) + \gamma_c(G_2) + \dots + \gamma_c(G_{t+1}))$
$= q(SL_{\frac{(t+1)(t+2)}{2}})$.

Definition 3.3: Let R_1 = {v'_1, v'_2, …, v'_t} and R_2 = {v_1, v_2, …, v_t} be two vertex sets in SL_t such that v'_i's are pendant vertices and v_i's are degree 3. The degree splitting graph of Sunlet Graph denoted by DS(SL_t) is obtained from S_t by adding vertices w_1, w_2 and joining w_1 to each vertex of R_1 and w_2 to each vertex of R_2.

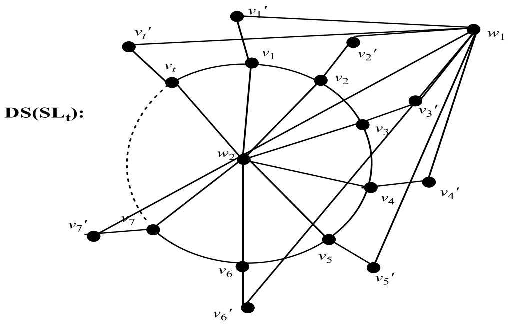

Figure 3.2: DS(SL_t)

Theorem 3.4: DS($SL_{\frac{t(t+5)}{2}+2}$) admits CDD(G_1, G_2, …, G_{t+2})

Proof: Let $V(SL_{\frac{t(t+5)}{2}+2}) = \{v_1, v_2, \ldots, v_{\frac{t(t+5)}{2}+2}, v_1', v_2', \ldots, v'_{\frac{t(t+5)}{2}+2}\}$. Then deg($v_i$) = 3 and deg($v_i'$) = 1, i = 1, 2, 3, …, $\frac{t(t+5)}{2}$ + 2. Let us take $R_1 = \{v_1', v_2', \ldots, v'_{\frac{t(t+5)}{2}+2}\}$ and $R_2 = \{v_1, v_2, \ldots, v_{\frac{t(t+5)}{2}+2}\}$. Adding vertices w_1 and w_2 to join each vertex of R_1 and R_2 respectively. Also deg(w_1) = deg(w_2) = Δ.

Consider the star $K_{1,\ \frac{t(t+5)}{2}+2}$ rooted at w_2. Take G_1 = $K_{1,\ \frac{t(t+5)}{2}+2}$. Let H = DS($S_{\frac{t(t+5)}{2}+2}$) − G_1. Then cycle C : $v_1\, v_2 \ldots v_{\frac{t(t+5)}{2}+2}\, v_1$ is in H. Let $e_i = v_iv_i'$ and $e_i' = v_iw_2$, i = 1, 2, …, $\frac{t(t+5)}{2}$ + 2. Now, $C = \bigcup_{k=1}^{t+1} P_{k+2}$. Then G_{k+1} = $P_{k+2} \cup \{\bigcup_i (e_i \cup e_i') / \frac{k(k+1)}{2} \le i \le \frac{k(k+3)}{2}\}$. Thus DS($SL_{\frac{t(t+5)}{2}+2}$) = $[\bigcup_{k=1}^{t+1} G_{k+1}] \cup G_1$. Hence DS($SL_{\frac{t(t+5)}{2}+2}$) can be decomposed into (G_1, G_2, …, G_{t+2}).

Definition 3.5: Let $u_1', u_2', \ldots, u_t'$ be the pendant edges of SL_t such that $v_iv_i' = u_i'$, i = 1, 2, 3, …, t and $u_1, u_2, \ldots, u_t$ be the edges such that $v_iv_{i+1} = u_i$, i = 1, 2, 3, …, t. Then the Line graph of Sunlet Graph is denoted by L(SL_t) such that u_1 is adjacent to u_2, u_t, u_1', u_2' and u_i is adjacent to u_{i+1}, u_{i-1}, u_i', u_{i+1}', i = 2, 3, …, t

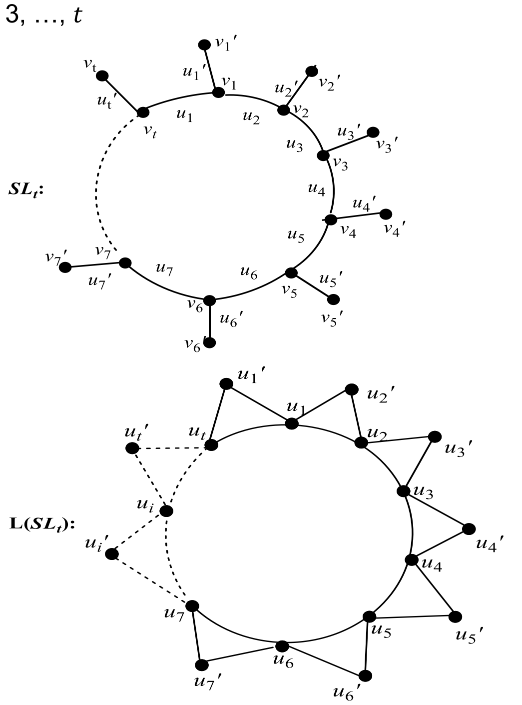

Figure 3.3: L(SL_t)

Theorem 3.6: L($SL_{\frac{t(t+5)}{2}+2}$) admits CDD(G_1, G_2, …, G_{t+1})

Proof: Let $G_1 = \{u_iu_{i+1} \cup u_iu_{i+1}' \cup u_{i+1}'u_{i+1} / 1 \le i \le 2\}$, $G_2 = \{u_iu_{i+1} \cup u_iu_{i+1}' \cup u_{i+1}'u_{i+1} / 3 \le i \le 5\}$, $G_3 = \{u_iu_{i+1} \cup u_iu_{i+1}' \cup u_{i+1}'u_{i+1} / 6 \le i \le 9\}$, …, $G_{t+1} = \{u_iu_{i+1} \cup u_iu_{i+1}' \cup u_{i+1}'u_{i+1} / \frac{t(t+3)}{2} + 1 \le i \le \frac{t(t+5)}{2} + 2\}$. Then $\gamma_c(G_1)$ = 1, $\gamma_c(G_2)$ = 2, …, $\gamma_c(G_{t+1}) = t + 1$. Therefore, L($SL_{\frac{t(t+5)}{2}+1}$) = $G_1 \cup G_2 \cup \ldots \cup G_{t+1}$. Hence L($SL_{\frac{t(t+5)}{2}+2}$) can be decomposed into (G_1, G_2, …, G_{t+1}).

Remark 3.7: L($SL_{\frac{t(t+5)}{2}+1}$) admits CDD(G_1, G_2, …, G_{t+1})

Theorem 3.8: If L($SL_{\frac{t(t+5)}{2}+2}$) admits CDD(G_1, G_2, …, G_{t+1}), then $3\sum_{i=1}^{t+1} \gamma_c(G_i) = q(\mathrm{L}(SL_{\frac{t(t+5)}{2}+2})) - 3(t+1)$.

Proof: Since $\gamma_c(G_i) = i$, i = 2, 3, 4, …, t + 1, i vertices form a minimum connected dominating set of G_i. Each G_i, i = 2, 3, 4, …, t + 1 consist of i + 1 edge disjoint union of triangles. Therefore,

$$q(G_i) = 3\,(i + 1)$$
$$= 3\,(\gamma_c(G_i) + 1),\ i = 2, 3, 4, \ldots, t + 1.$$

Now,

$$3\sum_{i=1}^{t+1} \gamma_c(G_i) = 3\,(\gamma_c(G_1) + \gamma_c(G_2) + \ldots + \gamma_c(G_{t+1}))$$
$$= q(G_1) - 3 + q(G_2) - 3 + \ldots + q(G_{t+1}) - 3$$
$$= q(G_1) + q(G_2) + \ldots + q(G_{t+1}) - 3(t+1)$$
$$= q(\mathrm{L}(SL_{\frac{t(t+5)}{2}+2})) - 3(t+1).$$

Remark 3.9: If L($SL_{\frac{t(t+5)}{2}+1}$) admits CDD(G_1, G_2, …, G_{t+1}), then $3\sum_{i=1}^{t+1} \gamma_c(G_i) = q(\mathrm{L}(SL_{\frac{t(t+5)}{2}+1})) - 3t$.

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