Multi-Criteria Optimization Techniques In DEA: Methods And Its Applications

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Abstract: This research article explores on multi-objective optimization methods namely multi-objective model optimization, global criterion method, bounded objective function method, lexicon-graphic method etc. The multiplier problem proposed by Charnes, Cooper and Rhodes (CCR) problem has been discussed in this paper. Own efficiency and cross efficiency of DMU's are defined here. Aggressive formulation of cross efficiency and benevolent formulation of cross efficiency have been proposed in this article. Besides MCDEA and MOLP problems are presented. The global efficiency approach of a multi objective model and the concept of super efficiency which works as a tool to develop the discriminating power of DEA in the presence of efficiency DMU's are depicted here.

Keywords: MCDEAP, MOLPP, DEA, DMU, peer count, own efficiency, cost efficiency, super efficiency, global efficiency.

1. INTRODUCTION

Multi-criteria models have more than one objective function such that these object to functions have to minimize simultaneously subject to a common set of constraints. Taking one objective function at a time, one can optimize it subject to a common set of constraints. The optimal solution of one objective need not be optimal for another. Since the objective functions are conflict in nature this would happen. There are two prominent approaches to solve a multiple objective model. The first approach is Pre-emptive and other is based on weighted sum of objective functions. In the former case the objective functions are arranged according to their importance. Taking into consideration the highest priority objective function it is optimized subject to all constraints. The optimal solution gives the optimal value of the objective function. We select the second highest priority objective function and optimize it subject to all the constraints of the multi-objective model and introduce one constraint extra. This constraint is imposed such that the search for optimal solution should not degrade the optimal value of the highest priority problem. In this way one solves as many optimization problems as there are multi-criteria objective functions. After one objective function has been optimized in pre-emptive processing of a multi criteria model, solutions obtained in the sub- sequent stages turnout to be alternative optima in the first, That is pre-emptive optimization places very great emphasis on the first objective, with all later steps limited to alternative optima in the highest priority objective. This fact is not surprising because in pre-emptive approach to solve a multi-criteria optimization problem, in each stage we impose an additional constraint, consequently, the solution space contracts.

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Every feasible solution of a later stage is feasible solution of the former stage. Let us consider a dual criteria optimization problem, for which we can construct an efficiency frontier. Out of the two objective functions, we can force one objective function to be a constraint that depends on a parameter. Varying the values of θ on an appropriately formulated range the other objective function can be optimized. Measuring the value of θ along one axis, say the horizontal axis and the optimal values of the objective function along vertical axis the piece wise linear efficiency frontier can be obtained. A multi-criteria model possesses numerous efficient solutions; the solutions space is not empty. The priority of efficiency is based on pare to optimality. A feasible solution to a multi objective optimization model is said to be an efficient of the feasible solution space if no other feasible solution scores at least as well in all objective functions and strictly better in one. Inferior feasible solutions are said to be dominated by efficient solutions. Andrea Kaim .et.al, in 2018, in their research article proposed a review of optimization techniques for land use allocation problems and depicted a crystal clear root map for choosing optimization techniques. In 2015 Xiaoya Ma et.al. In their paper described an LUA model constructed on MOAIOA to get the foremost sustainable MOLUA optimization solution. Guadalupe Azuara Garcia, et.al. in 2017 in their research article proposed a multi-objective model for sustainable land use allocation in which land use allocations are generated and NSGA-II has been reshaped.

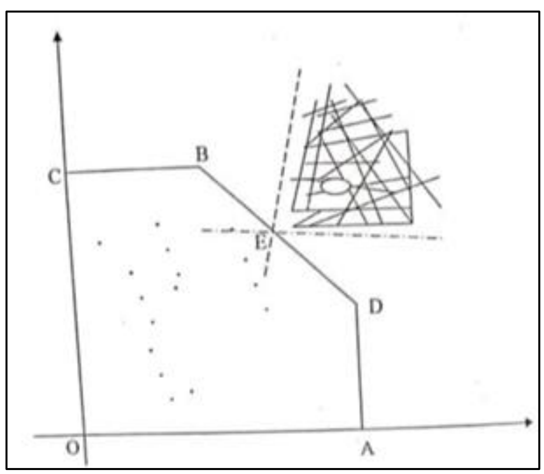

Fig: (1)

In the above figure the solution space bound by the line segments BC, OA, AD and OC of a multi criteria optimization problem is convex. Two dotted lines pass through the point E that lies on the line segment BD are objective functions. The shaded area partially bound by these lines gives solutions that are better than those in the shaded area. But, none, of them is feasible for the multi-objective model optimization problem. E is an efficient. For an efficient point no distinct feasible solution lies in the region bound by the objective function contours passing through this point where each point of this region gives a better solution than the efficient point of the multi-objective model optimization problem.

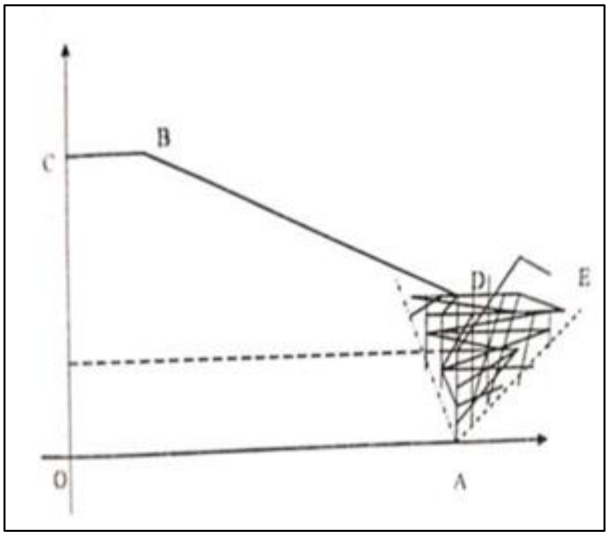

Fig: (2)

The objective function contours pass through the point A of feasible solution space. The solution space represented by the shaded region consists of solutions better than A. But some of these solutions are feasible and some other is inferior. For example, the feasible solution at E is better than A. Hence, A is dominated by the feasible solution at E. weighted sum of objectives of a multi-objective model optimization leads to a single objective function such that the weight associated with maximizing objective function is positive and that of minimizing objective function is negative, if the objective is maximization.

2. MULTI-OBJECTIVE MODEL OPTIMIZATION:

Mathematically, a multi-criteria optimization problem may be expressed as,

Minimize $f_1(x), f_2(x), \dots \dots \dots \dots \dots, f_n(x)$

Subject to $g_j(x) \leq 0,\ j = 1,2,3, \dots \dots \dots .., k$

There are n objective functions that are to be minimized simultaneously subject to set of$k$ inequalityconstraints. The contours of objective functions and /or the constraints may be non-linear. In general a feasible solution vector that minimizes the objective functions simultaneously cannot exist. However, pare to optimal solutions exist for a majority of multi-criteria models. A feasible solution vector X is said to be Pareto optimal, if there does not exist a solution y such that,

$(i) f_i(y) \leq f_0(x), i = 1,2, \dots \dots \dots \dots .., n$

(ii) $f_i(y) \leq f_0(x)\ for\ some\ i$

Several methods were proposed to solve multi objective model problems.

3. GLOBAL CRITERION METHOD: A GLOBAL CRITERION IS FORMULATED AS FOLLOWS

$$F(X) = \sum_{i=1}^{n} \left[\frac{f_i(x_i^*) - f_i(x_i)}{f_i(x_i^*)}\right]$$

$F(X)$is minimized subject to the constraints,

$g_j(X) \leq 0, for\ j = 1,2, \dots \dots \dots, k$

Thus, the problem is expressed as

$F(x) = \min_X \{F(X): g_j(X) \leq 0, j = 1,2, \dots \dots .. k\}$

x_i^* Refers to optimal solution. For any x_i,

$f_i(x_i^*) - f_i(x_i)$ measures the deviation of the objective function from ideal value.

4. BOUNDED OBJECTIVE FUNCTION METHOD:

In the bounded objective function method the minimum and maximum acceptable achievement levels are specified for each objective function.

$$L_i \leq f_i \leq V_i$$

The optimal solution is found minimizing the objective function of top most priority. If r^{th} objective function is of top most priority, then we solve the following optimization problem:

$Min\ f_r(x)$

Subject to $g_j(x) \leq 0, for\ j = 1,2, \dots \dots \dots, m$

$L_i \leq f_i(x) \leq V_i\ \ i = 1,2, \dots \dots \dots \dots . n,\ \ i \neq r$

5. LEXICO GRAPHIC METHOD:

In this method the objective functions are assigned with priorities depending upon their importance. Suppose $f_1(x), f_2(x), \dots \dots f_n(x)$ be the objective functions expressed according to their priority.

(i) $Min\ f_1(x)$

Subject to $g_j(x) \leq 0, for\ j = 1,2, \dots \dots \dots, m$

Let x^* be the optimal solution for this problem. The optimal value of the objective function is $f_1(x_i^*)$

(ii) Solve the following optimization in the second stage:

$Min\ f_2(x)$

Subject to $f_1(x) = f_1(x_1^*)$ an

$g_j(x) \leq 0, for\ j = 1,2, \dots \dots \dots, m$

Let x_2^* be the optimal solution of the problem then $Min\ f_2(x)$ = $f_1(x_2^*)$

In this process at n^{th} stage we solve $Min\ f_n(x)$

Subject to

$f_i(x) = f_i(x_i^*), for\ i = 1,2, \dots \dots . n - 1\ and$

$g_j(x) \leq 0, for\ j = 1,2, \dots \dots \dots, m$

The optimal solution of this problem is the optimal solution of the multi-objective programming Problem.

6. DEA-GLOBAL EFFICIENCY:

The multiplier problem proposed by Charnes, Cooper and Rhodes (CCR) problem can be expressed of the form.

$h_0 = Max\ h = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}}$

Subject to $\frac{\sum_{r=1}^{s} u_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1, for\ j = 1,2, \dots .. n\ and\ u_r, v_i \geq 0$

The suffix '0' refers to the Decision Making Units (DMU), whose efficiency is under evaluation.

$y_{rj}: r^{th}\ output\ by\ j^{th}\ DMU\ \ and$

$x_{ij}: i^{th}\ input\ employed\ by\ j^{th}\ DMU$

The numerator and denominator of the objective function refer to the 'Virtual Output' and 'virtual input' respectively as such ratio refers to virtual output per unit of virtual input foregone. u_r and v_i are outputs and input multipliers, which are unknown. The multipliers are not known a priori, but determined by solving an optimization problem. These multipliers are expected to be non-negative. The DMU under evaluation chooses the multipliers to the best of its advantage forcing virtual output to virtual input ratios of all DMUs including its own ratio not to be exceeding unity. Exploying the Charness, Cooper transformation the above FPP is reduced to a LPP expressed as follows:

$Max\ \sum_{r=1}^{s} u_r y_{r0}$

Subject to $\sum_{i=1}^{m} v_i x_{i0} = 1,\quad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, for\ j = 1,2,\ldots\ldots\ldots, n\ and\ u_r, v_i \ge 0$

The CCR problem accounts constant RTS only.

Own efficiency Cross efficiency: The linear programming problem solved for DMU_0 presents output and input weights to the best of its advantage. The efficiency of DMU_0computed in this procedure is known as own efficiency. Armed with u_{ro}, v_{i0} (output and input weights of DMU_0) DMU_0 evaluates other DMUs. For example DMU_j as follows

$$h_j = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_{i0} x_{ij}}$$

$h_j$is viewed as cross efficiency of DMU_j evaluated by DMU_0 with this chosen weights. It is possible that some DMU_s emerge to be technically the most efficient. For each one of them efficiency score is unity. For further discrimination of these DMU_s and to accredit them, one can use the cross efficiency matrix. If j^{th} DMU is efficient in the standards of its own $\frac{\sum_{j\ne k} h(j,k)}{n-1}$ measures mean cross efficiency of DMU_j. Mean cross efficiency reveals how DMU_i is evaluated by its peers. A decision making unit which is efficient by its standards and whose mean peer appraisal efficiency is the largest may be assigned with rank one. Thus, cross efficiencies can be used for further discriminating the efficient DMU_s. If DMU_k is efficient and the optimal weights are not unique, so that the k^{th} column of cross efficiency matrix is generated by the first optimal solution of DMU_k. We can use the aggressive formulation or the benevolent formulation of cross efficiencies and look for unique multiplier estimates. The cross efficiency assigned to DMU_j by DMU_k may be expressed as

$$h(j,k) = \frac{\sum_{r=1}^{s} u_{rk}^{k} y_{rj}}{\sum_{i=1}^{m} v_i^k x_{ij}}, j \ne k$$

If $j = k, then\ h(k,k)$ represents self-appraisal efficiency of DMU_k .we propose the following cross efficiency matrix

Rated DMU

	1	2	…	k	…..	n
1	h(1,1)	h(1,2)		h(1,k)		h(1,n)
2	h(2,1)	h(2,2)		h(2.k)		h(2.n)
.	…	…		…		…
j	h(j,1)	h(j,2)		h(j, k)		h(j, n)
	…	…		…		…
n	h(n,1)	h(n,2)		h(n, k)		h(n, n)

$h(j,j)$: Self -appraisal efficiency of DMU_i and $h(j,k)$: Peer appraisal efficiency of DMU_i by DMU_k. A DMU which is efficient by self-appraisal need not be so by peer appraisal. $h(j,k)$ is the efficiency of DMU_i rated by DMU_k . Suppose the CCR problem is solved n times, one time for one decision making unit and the cross efficiency matrix is obtained.

$H(j) = \frac{\sum_{j\ne k} h(j,k)}{n-1}$ and $H(j)$ measures DMU_j's averaged appraisal of its peers against which j compares itself.

7. AGGRESSIVE FORMULATION OF CROSS EFFICIENCY:

The input and output weights of multiplier problem in which DMU maximizes virtual output per unit of virtual input foregone are not unique since it is likely that the self-appraisal problem possesses multiple optimal solutions. The weights displayed are the weights of the optimal solution that the computer picks up the earliest. One means to get away from the ambiguity is to introduce a secondary objective function so that we formulate and solve a goal programming problem with duals.

Goal-1: Maximize simple efficiency of DMU_k
Goal-2: Minimize the order DMU_s cross efficiencies in some manner

The final solution of the goal programming problem leads to the specification of not only unique but also non-zero weights. The goal programming problem is called an aggressive formulation. Benevolent formulation of cross efficiency: The benevolent formulation of cross efficiency is a procedure to formulate a goal programming problem with the primary goal to maximize DMU_k's simple efficiency, the secondary goal is to maximize the cross efficiencies of other DMUs. Thus, to further discriminate the decision making units we can use either aggressive or benevolent formulation.

Primary goal:

$$h_k = Max\ h = \frac{\sum_{r=1}^{s} u_{rk}^{k} y_{rk}}{\sum_{i=1}^{m} v_i^k x_{ik}}$$

Subject to $\frac{\sum_{r=1}^{s} u_{rk}^{k} y_{rj}}{\sum_{i=1}^{m} v_i^k x_{ij}} \le 1, j = 1,2,\ldots\ldots\ldots\ldots n\ and\ u_r^k, v_i^k \ge 0$

By CCT the ratio problem is reduced to a LPP.

$h_k = Max\ h = \sum_{r=1}^{s} u_{rk}^{k} y_{rk}$

Subject to $\sum_{i=1}^{m} v_i^k x_{ik} = 1$

$$\sum u_r y_{kj} - \sum u_i x_{ij} \le 0, j = 1,2,\ldots\ldots..n \quad and\ u_r^k, v_i^k \ge 0$$

Let $H_k = \sum_{\substack{j=1\\ j\ne k}}^{n} \frac{h(j,k)}{n-1}$ and $F_k = \sum_{\substack{j=1\\ j\ne k}}^{n} \frac{h(k,j)}{n-1}$

Where H_k measures average appraisal by peers of DMU_k and $F_k$measures average appraisal by peers of DMU_k

Secondary goal: Thek^{th} DMU maximizes F_k under benevolent formulation of cross efficiency while it minimizes F_k under aggressive formulation of cross efficiency.

$$F_k = \sum_{\substack{j=1\\ j\ne k}}^{n} \frac{h(k,j)}{n-1} = (n-1)F_k \sum_{j\ne k}^{n} h(k,j) = \sum_{j\ne k}^{n} \left[\frac{\sum_{r=1}^{s} u_r^k y_{rj}}{\sum_{i=1}^{m} v_i^k x_{ij}}\right]$$

The objective function $(n-1)F_k$ is non-linear which cannot be solved by standard linear programming techniques. We replace the non-linear objective function by the following.

$$F_k^1 = \sum_{j\ne k}^{n} \left\{\sum_{r=1}^{s} u_r^k y_{rj}\right\} - \sum_{j\ne k}^{n} \left\{\sum_{i=1}^{n} v_i^k x_{ij}\right\}$$

$$F_k^1 = \sum_{r=1}^{s} u_r^k \left\{\sum_{j\ne k}^{n} y_{rj}\right\} - \sum_{i=1}^{m} v_i^k \left\{\sum_{j\ne k}^{n} x_{ij}\right\}$$

$F_k^1$is linear in multipliers, its minimization or maximization refers to aggressive or benevolent formulation of cross efficiency. Another reformulation of goal (2) is as follows

$$(n-1)z_k = \frac{\sum_{j\ne k}^{n} \sum_{r=1}^{s} u_r^k y_{rj}}{\sum_{i=1}^{n} v_i^k x_{ij}} = \frac{\sum_{r=1}^{s} u_r^k \{\sum_{j\ne k}^{n} y_{rj}\}}{\sum_{i=1}^{m} v_i^k \{\sum_{j\ne k}^{n} x_{ij}\}}$$

By CCT the FPP is changed into a LPP.
Aggressive formulation:

$$Min\ Z_k^1 = \sum_{r=1}^{s} u_r^k \left\{ \sum_{j \neq k}^{n} y_{rj} \right\}$$

Subject to $\sum_{r=1}^{m} v_i^k \{\sum_{j\neq k}^{n} x_{ij}\} = 1$

$\sum_{r=1}^{s} u_r^k y_{rj}$ - $\sum_{i=1}^{m} v_i^k x_{ij} \leq 0, j \neq k, j = 1,2, \dots \dots \dots n\ and\ u_r^k, v_i^k \geq 0$

Benevolent formulation: $Max\ Z_k^1 = \sum_{r=1}^{s} u_r^k \{\sum_{j\neq k}^{n} y_{rj}\}$

Subject to $\sum_{r=1}^{m} v_i^k \{\sum_{j\neq k}^{n} x_{ij}\} = 1$

$\sum_{r=1}^{s} u_r^k y_{rj}\ \ \sum_{i=1}^{m} v_i^k x_{ij} \leq 0, , j = 1,2, \dots \dots \dots n\ and\ u_r^k, v_i^k \geq 0$

(i) Average cross efficiency is used to distinguish among the 100% efficient DMUs, establishing a meaningful ranking.
(ii) CE is applied to recognize maverick decision making units. A maverick decision making unit enjoys the greatest relative increment when it moves from peer –appraisal to self-appraisal. For DMU_K the maverick index is given as,

$$M_k = \frac{h(k,k) - H_k}{n-1}$$

The higher $M_{k,}$ the more of a maverick is DMU_k. The maverick index can be calculated to each DMU in competition. The other end of index M_k identifies the all-round performers.

8. MULTIPLE CRITERIA APPROACH-

DEA: Let d_0 be the slack associated with the constraint.

$\sum_{r=1}^{s} u_r y_{r0} - \sum_{i=1}^{m} v_i x_{i0} \leq 0 \Rightarrow \sum_{r=1}^{s} u_r y_{r0} - \sum_{i=1}^{m} v_i x_{i0} + d_0 = 0$

$d_0 = 0 \Leftrightarrow \sum u_r y_{r0} - \sum v_i x_{i0} = 0 \Leftrightarrow \frac{\sum u_r y_{r0}}{\sum v_i x_{i0}} = 1$

$\Leftrightarrow$ DMU_0 is 100% technical efficient. DMU_0 is technically efficient, if and only if $d_0 = 0$

The following linear programmes are equivalent.

$Min\ d_0$

Subject to $\sum v_i x_{i0} = 1$

$\sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 0, for\ j = 1,2, \dots \dots \dots ..n$ (8.1)

$\sum u_r y_{r0} - \sum v_i x_{ij} + d_j = 0$ and $u_r, v_i \geq 0$

$Max\ h_0 = \sum u_r y_{r0}$

Subject to $\sum v_i x_{i0} = 1$ (8.2)

$\sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 0, for\ j = 1,2, \dots \dots \dots ..n$ and $u_r, v_i \geq 0$

The form of the multiple criteria DEA model is not unique. It depends on the efficiency criteria used. The MCDEA problem that has the criteria
(i) Min d_0 (ii) Minimize the maximum deviation (iii) Minimize the sum of the deviation.
Consider the following optimization problems:

$Min\ d_0$

Subject to $\sum v_i x_{i0} = 1$
(8.3)

$\sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 0, for\ j = 1,2, \dots \dots \dots ..n$

and $u_r, v_i, d_j \geq 0$

A feasible solution of is a feasible solution of (8.3) is a feasible solution of (8.2)

Let $\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{d} \end{bmatrix}$ be a feasible solution of (8.2)

Define $\widehat{M} = \sum_{i \leq j \leq n}(d_j) \Rightarrow \widehat{M} \geq d_{j,} \forall j \ \Rightarrow\ \widehat{M} - d_j \geq 0$

$\begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{d} \end{bmatrix}$ is a feasible solution of (8.3)

Thus, the two linear programming problems (8.2) are equivalent. The optimal values of d_0 implied by the two problems are also one and same.
Consider the following LPP:

$Min\ M$

Subject to $\sum_i v_i x_{i0} = 1$ and

$\sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 1$ (8.4)

$M - d_j \geq 0\ and\ u_r, v_i, d_j \geq 0, \forall\ r, i\ and\ j$

Between (8.3) and (8.4) the constraints are one and the same.
The technical efficiency implied by (8.4) is less that implied by (8.3)

$M \geq d_{j\ ,} j = 1,2, \dots .n\ M \geq \underset{i \leq j \leq 1}{Min}(d_j) \geq d_0 and\ M \geq d_0$

$\Rightarrow Min\ M \geq Min\ d_0 \Rightarrow Min\ M \leq -Min \leq d_0$

$-M_0 \leq -\widehat{d_0} \Rightarrow 1 - M_0 \leq 1 - \widehat{d_0}$

Consider the following LPP

$Min\ \sum_{j=1}^{n} d_j$

Where $\sum v_i x_{i0} = 1$ and $\sum u_r y_{r0} - \sum v_i x_{i0} + d_j = 0$ (8.5)

$M - d_j \geq 0\ and\ u_r, v_i, d_j \geq 0, \forall\ r, i\ and\ j$

(8.2). (8.3) and (8.4) together represent a multiple objective linear programming. In an MOLP problem, it is generally impossible to find a solution that optimizes all objectives simultaneously. The real task in this situation is to find non-dominated solutions and to help select a most preferred one. A solution represented by a point in decision variable space is said to be non-dominated, if it is not possible to move the point within the feasible region to improve an objective function value without deteriorating at least one of the other objectives. A non-dominated solution is also called an efficient solution. A non-dominated solution set for an MOLP problem will always contain the optimal solution of optimization problems solved with only one objective. In an attempt (8.4), suppose the optimum solution is as follows:

$M_0 = Min\ M$

$u_r = u_r^*, v_i = v_i^*, d_0^* = 0$

Since M stands for maximum deviation we call (8.4) mini-max problem and DMU_0as mini-max efficient. If$d_0^* > 0$ then DMU_0 is mini-max inefficient. Consider the following linear programming problem:

$Min\ \sum_{j=1}^{n} d_j$

Subject to $\sum v_i x_{i0} = 1$, $\sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 0$ and $u_r, v_i, d_j \geq 0$ (8.6)

The optimization problem (8.6) is 'min sum' problem. The multiplier space is name for (8.3), (8.4) and (8.5) DMU_0 is min sum efficient if and only if $d_0^* = 0$ in the optimal solution of (8.5)

$\sum d_j^* = 0 \Rightarrow d_0^* = 0$ and $d_0^* = 0 \nRightarrow \sum d_j^* = 0$

The value of $d_0^* = 0$ may assume different values under different criteria used as objective function.

$M_0 = 0 \Rightarrow d_0^* = 0$ and $d_0^* = 0 \nRightarrow M_0 = 0$

$M_0 = 0 \Leftrightarrow \sum d_j^* = 0$

The mini-max and mini sum criteria do not give most favourable weights to the multipliers of the DMU under evaluation. Efficiencies implied by mini max and min sum criteria are more restrictive than the technical efficiency of the standard DEA multiplier problem. It is more difficult to achieve min-max or min sum efficiency than to achieve classical DEA efficiency. The above discussion reveals that the mini-max or min sum criterion yields fewer efficient DMU_s. To discriminate efficient decision making units like aggressive and benevolent formulations the min-max and min sum criteria can be implemented. A DMU that is self-rated inefficient may turn out to be peer rated efficient when the objective is minimizing the sum of all deviational variables.

Global efficiency: Data Envelopment analysis MOFPP is given as follows:

$Max \left\{\frac{\sum u_r y_{rj}}{\sum v_i x_{ij}}, j = 1,2, \dots \dots . n\right\}$ (8.7)

Subject to $\frac{\sum u_r y_{rj}}{\sum v_i x_{ij}} \leq 1 \; and \; u_r, v_i \geq 0$

Optimization of (8.7) n times, one time for one decision making unit presents the OE point of (8.7), GPP with n objectives.

Let $h_j^* = \frac{\sum u_r^* y_{rj}}{\sum v_i^* x_{ij}}$ and optimal efficiency point: $(h_1^*, h_2^* \dots \dots \dots . h_n^*)$

In classical DEA approach the self-rated DEA uses input and out weights that are most favourable to it in an attempt to evaluate its technical efficiency. One fractional programming problem is solved for one decision making unit. Consequently, the input and output weights change from one decision making unit to another. The DEA approach places a DMU into one of the two mutually exclusive sets of efficient DMU_s. The method has no power in further discriminating the efficient DMU_s in competition. To much flexibility in the choice of weights leads many DMU_s to emerge as efficient. The global efficiency approach provides a multi-objective model that gives common multiplier weights for all DMU_s in such a way that the resultant own efficiencies are as close as possible to the ideal point $(h_1^*, h_2^* \dots \dots \dots . h_n^*)$ in some sense.

Consider the following model:

$Min \; \theta \, \bar{d} + (1-\theta) z$

Subject to $\frac{\sum u_r y_{rj}}{\sum v_i x_{ij}} + d_j = h_j^*$ (8.8)

$z - d_j \geq 0 \; and \; u_r \geq 0, v_i \geq 0$ Where $\bar{d} = \sum_{j=1}^{n} \frac{d_j}{n}$, $d_j = j^{th}$ deviational variable

$\frac{\sum u_r y_{rj}}{\sum v_i x_{ij}}$ =Adjusted global efficiency score of j^{th} DMU, whose value is known. The real number d_j stands for departure of j^{th} DMU_s efficiency score from global efficiency score. The optimal solution of (8.8) provides efficiency scores of various DMU_s, with common weights. If $\theta = 1$, the problem (8.8) reduces to a model in which mean of deviational is minimized, consequently, the objective and objective function are $Min \; \bar{d}$. If $\theta = 0$, the objective and objective function reduces to the following:

Minimize z, which is equivalent to minimize the maximum deviation.

Varying θ between 0 and 1 the model provides flexibility to obtain the common weights in different sets, one set for one value of θ.

(i) Consider the in equation $\frac{\sum u_r y_{rj}}{\sum v_i x_{ij}} \leq h_j^*$

$$\Leftrightarrow \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \leq 0$$

(ii) $z - d_j \geq 0 \Leftrightarrow d_j - z \leq 0$, where $d_j = h_j^* - \frac{\sum u_r y_{rj}}{\sum v_i x_{ij}}$

$$z - d_j \geq 0 \Leftrightarrow h_j^* - \frac{\sum u_r y_{rj}}{\sum v_i x_{ij}} - z \leq 0$$

The optimization problem (8.8) for $\theta = 0$ can be equivalently expressed as

$Min \; z$

Subject to $\sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \leq 0$

$h_j^* - \frac{\sum u_r y_{rj}}{\sum v_i x_{ij}} - z \leq 0 \; and \; u_r, v_i, z \geq 0.$ (8. 9)

The optimization problem ( ) is non-linear

Stage (1) of multi-criteria problem:

For each decision making unit solve the following multiplier problem

$h_0^* = Max \sum u_r y_{r0}$

Subject to $\sum v_i x_{i0} = 1, \sum u_r y_{rj} - \sum v_i x_{ij} \leq 0, j = 1,2, \dots \dots \dots . n$

If there are n DMU_s the LPP (8.3) is solved n times, one time for one decision making unit. The efficiency score of j^{th} denoted by h_j^*. Let $(h_1^*, h_2^*, \dots \dots \dots \dots . h_n^*)$ be ideal point of optimization of n decision making units in competition.

Stage (1) of multi-criteria problem: Consider the constraints

$$\frac{\sum u_r y_{rj}}{\sum v_i x_{ij}} \leq h_j^* \Leftrightarrow \frac{\sum u_r y_{rj}}{h_j^* \sum v_i x_{ij}} \leq 1 \Leftrightarrow \sum u_r y_{rj} \leq h_j^* \sum v_i x_{ij}$$

$$\Leftrightarrow \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \leq 0$$

$\Leftrightarrow \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} + d_j = 0$ (8.10)

where $d_j \geq 0$ is the deviation variable of the constraint of j^{th} DMU.

If z refers to maximum deviational variable we formulate additional constraints.

$z - d_j \geq 0 \; and \; z + \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \geq 0$ (8.11)

with the constraints (1) and (2) in mind we can propose a number of linear programming problems.

(1) $Min \; d_0$

Subject to

$\sum v_i x_{i0} = 1$, $\sum u_r y_{rj} - h_j^* \sum v_i x_{ij} + d_j$ (8.12)

$z + \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \geq 0$ and $u_r, v_i, z \geq 0$

If $d_0 = 1$ then DMU_0 is efficient, otherwise inefficient. Consider the linear programming problems (8.3) and (8.12) . A DMU that is efficient if (8.2) is solved need not be efficient if (8.12) is solved. Consequently, lesser number of DMU_s emerge to be efficient in (8.12) rather than in (8.2).

$\sum u_r y_{rj} - \sum v_i x_{ij} \geq 0 \Rightarrow \sum u_r y_{rj} \geq \sum v_i x_{ij} \geq h_j^* \sum v_i x_{ij}$

Since $h_j^* \leq 1 \Rightarrow \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \geq 0$

Let, $S_1 = \{(u, v, z, d_j): \sum v_i x_{i0} = 1,$
$\sum u_r y_{rj} - h_j^* \sum v_i x_{ij} + d_j = 0,$
$z + \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \geq 0, u, v, z, d_j \geq 0\}$ and
$S_2 = \{(u, v, z, d_j): \sum v_i x_{i0} = 1, \sum u_r y_{rj} - \sum v_i x_{ij} + d_j = 0, z + \sum u_r y_{rj} - \sum v_i x_{ij} \geq 0,$
$u, v, z, d_j \geq 0\}$

$S_2 \subseteq S_1$

$Min\{d_0 : d_0 \in s_2\} \geq Min\{d_0 : d_0 \in s_1\}$

Let $Min\{d_0 : d_0 \in s_2\} = \hat{d}_0$ $and \; Min\{d_0 : d_0 \in s_1\} = \widetilde{d_0}$

$\hat{d}_0 \geq \widetilde{d_0} \Rightarrow \widetilde{d_0} = 1 \; then \; \hat{d}_0 \geq 1 \Rightarrow \hat{d}_0 = 1.$

Model (2)

$Min \; z$

Subject to $\sum v_i x_{i0} = 1, \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} + \widetilde{d}_j = 0$

$z + \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} \geq 0$ and $u_r, v_i z, \widetilde{d}_j \geq 0$ (8.13)

Let $d_j \; and \; \widetilde{d}_j$ be the deviational variable of the linear programming problems (8.5) and (8.13) respectively

Let $d_j \geq \widetilde{d}_j, j = 1,2, \dots \dots \dots \dots \dots \dots . . n$

$Max\{d_j\} \geq Max\{\widetilde{d_0}\}$

$1 \leq j \leq n 1 \leq j \leq n$

$M \geq Max\{d_j\}$, $Min\{M : M \in s_2\} = M^* = \max_j\{d_j\}$

$Min\{z : z \in s_1\} = z^* = \max_j\{\widetilde{d}_j\}$

$Min\{M : M \in s_2\} \leq Min\{z : z \in s_1\}$

Model (3)

$Min \; \frac{1}{n}\left(\sum d_j\right)$

Subject to

$\sum v_i x_{i0} = 1, , \sum u_r y_{rj} - h_j^* \sum v_i x_{ij} + \widetilde{d}_j = 0$ (8.14)

$z + \sum u_r\, y_{rj} - h_j^* \sum v_i\, x_{ij} \geq 0$

Let d_j and $\widetilde{d_j}$ be the deviational variable of the linear programming problems (8. 6) and ( ) respectively

$d_j \leq \widetilde{d_j}\ ,\forall j \ \Leftrightarrow\ \frac{1}{n}\left(\sum \widetilde{d_j}\right) \leq \frac{1}{n}\left(\sum d_j\right)$

$Min\left(\frac{1}{n}\sum \widetilde{d_j}\right) \leq Min\left(\frac{1}{n}\sum d_j\right)$

9. PEER COUNT- DISCRIMINATING POWER OF DEA

When a classical DEA problem is solved any DMU of the choice turns out be efficient or inefficient. For efficient DMU_s efficiency score is one and for inefficient DMU_s in terms of their returns to scale, the following multiplier problem may be solved:

$Max\ h_j^* = \sum u_r y_{r0} + w$

Subject to

$\sum v_i x_{i0} = 1, \sum u_r\, y_{rj} - h_j^* \sum v_i\, x_{ij} \leq 0\ and\ u_r, v_i\ \geq 0$ (8.15)

In the optimal solution of the above problem sign of w reveals the nature of the returns to scale, increasing or constant or decreasing.

$w^* < 0$ ⇔RTS are increasing

$w^* = 0$ ⇔RTS are constant

$w^* > 0$ ⇔ Returns to scale are decreasing

The classical CCR multiplier problem is given by

$Max\ h_j^k = \sum u_r y_{r0}$

Subject to

$\sum v_i x_{i0} = 1,\ \sum u_r\, y_{rj} - \sum v_i\, x_{ij} \leq 0\ and\ u_r, v_i\ \geq 0$ (8.17)

In DEA LP problem if more and more decision variables are added, more and more DMU_S emerge to be efficient. To improve discriminatory power of DEA the problem ( ) is solved for all DMU_s such that one problem for one DMU and efficient DMU_S are defined. A DMU evaluated efficient by model one need not be efficient, if model (1), model (2) is solved, fewer of the memerge to be efficient. For further discrimination of efficient, DMU_s of model (2), we depend on peer count. The efficient DMU_S are leaders and the inefficient DMU_S are followers. The Dual problems of (8.18) and (8.19) are called the envelopment problems. These are formulated as follows.

$\lambda_1 = Min\ \lambda$

Subject to

$\sum_{i=1}^{n} \lambda_j x_{ij} \leq\ \lambda\ x_{i0},\ \ i = 1,2, \dots \dots \dots \dots m$ (8.18)

$\sum_{i=1}^{n} \lambda_j u_{rj} \geq u_{r0},\ \ r = 1,2, \dots \dots \dots \dots s$ and $\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0$

$\lambda_2 = Min\ \lambda$

Subject to

$\sum_{i=1}^{n} \lambda_j x_{ij} \leq\ \lambda\ x_{i0},\ \ i = 1,2, \dots \dots \dots \dots m$ (8.19)

$\sum_{i=1}^{n} \lambda_j u_{rj} \geq u_{r0},\ \ r = 1,2, \dots \dots \dots \dots s$ and $\sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0$

$Min\{\lambda: \sum \lambda_j x_{ij} \leq\ \lambda x_{i0},\ \sum \lambda_j u_{rj} \geq u_{r0}, \sum \lambda_j = 1, \lambda_j \geq 0\ \} \geq$

$Min\ \{\lambda: \sum \lambda_j x_{ij} \leq\ \lambda x_{i0},$

$\sum \lambda_j u_{rj} \geq u_{r0}, \lambda_j \geq 0\}$ and $\lambda_1 \geq \lambda_2$

In $\lambda_2$the pure technical efficiency is confounded with scale efficiency of the decision making unit whose efficiency is under evaluation. The LP problem (8.18) admits variable returns to scale while (8.19) admits constant returns to scale only. For an extremely efficient DMU, say DMU_0, the optimal solution combined with the constraints under constant returns to scale may be expressed as follows.

$\sum_j \lambda_j^*\, x_{ij} = x_{i0}, \forall i, \sum_j \lambda_j^*\, y_{rj} = y_{r0}, \forall r, \lambda_j^* = 0, \forall j \neq 0$
$and\ \lambda_j^* = 1$

If the DMU_0 is inefficient, the optimal solution combined with the constraints yield the following.

$\sum_j \lambda_j^*\, x_{ij} < x_{i0}, \sum_j \lambda_j^*\, u_{rj} \geq u_{r0}, \lambda_0^* = 0\ and\ \lambda_j^* \neq 1$, For at least one $j \neq 0$

The decision making units which correspond to the envelopment weights, not equal to zero are the peers of an inefficient DMU under evaluation. The peers of an inefficient DMU are one or more efficient DMU_s, such an efficient DMU that appears in the peer list of efficient DMU_s the largest number of times is the most efficient DMU efficient $DMJU_s$ under constant returns to scale formulation. However, possible ties may occur.

10. SUPER EFFICIENCY:

The union of 'super efficiency' serves as a tool to improve the discriminating power of data envelopment analysis, in the presence of efficiency DMU_s.

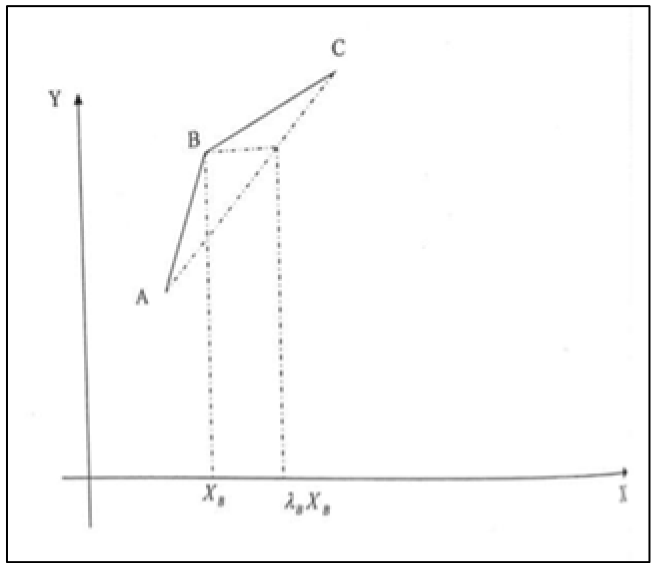

Fig: (3)

The above figure shows a part of the piecewise linear production frontier generated by DMU_s A, B and C. If the DMU B is removed from reference technology the production possibility set shrinks. The line segments AB and BC are removed from the frontier production function. Instead, in their place the line segment AC is added. For DMU B the following linear programming problem is solved.

$\lambda_B = Min\,\lambda, subject\ to\ \sum_{j=1, j\neq B}^{n} \lambda_j x_j \leq \lambda x_B, \sum_{j=1, j\neq B}^{n} \lambda_j y_j \leq u_B,$
$\lambda_j \geq 0, j \neq B\ and\ \lambda_B \geq 1$.

The efficiency score λ_B may be interpreted follows. To produce the output that is produced by DMU B, the remaining DMU_s viewed as a convex combination require using an additional input $(\lambda_B - 1)x_B$where$(\lambda_B - 1)x_B$ is the input saved by DMU B due to super efficiency. The LP problem postulated above is an input oriented problem. Similarly, we can propose an output oriented DEA problem.

$$\theta_B = Max\theta$$

$$\sum_{\substack{j=1 \\ j\neq B}}^{n} \lambda_j x_j \leq x_B, \sum_{j\neq B}^{n} \lambda_j y_j \leq \theta y_B\ ,\ \lambda_j \geq 0, j \neq B$$

DMU B is extremely efficient. Its input vector and output vector removed from the reference technology.

$0 \le \theta_B \le 1$. The efficiency score θ_B may be interpreted as follows. With the input vector X_B employed by DMU_B is $(1-\theta_B)Y_B$.

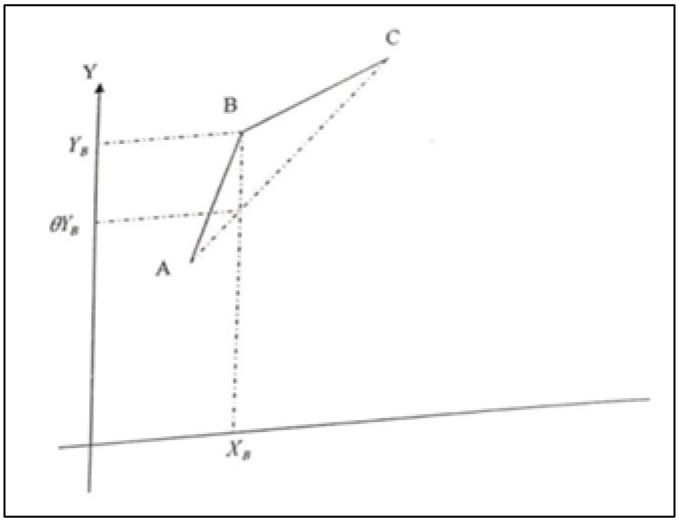

Fig: (4)

The production making units involved in competition can be divided into four groups. (i) Extremely efficient DMUs (ii) Efficient but not extremely efficient DMUs (iii) Weakly efficient DMUs (iv) Inefficient DMUs. The super efficiency problems are solved only for extremely efficient DMUs. The removal of a DMU that is nit extremely efficient from the reference technology will not bring any change to the production possible set. Consequently, the efficient scores remain to be the same.

11 CONCLUSION AND FUTURE RESEARCH:

In the above research paper a brief discussion on multi objective optimization methods has been done. Different types of multi objective models have been proposed. Moreover using discriminating power of DEA peer count is estimated. Finally this article presents the notion of super efficiency which serves as a tool to improve the discriminating power of DEA. In the context of future research one can estimate the super efficiency of extremely efficient DMUs and evaluate the different types of efficiency stability regions and their infeasibility in DEA.

12 REFERENCES

[1] Andrea Kaim, Martin Volk (2018)," A review of multi- criteria optimization techniques for agricultural land use allocation', Environmental Modelling and Software, Volume105, pages 79-83.

[2] Xiaoya Ma Xiang Zhao (2015),"Land use allocation based on a multi-objective artificial immune optimization model: An application in Anlu country, China", Sustainability, 2015, 7, 15632-15651.

[3] Guadalupe Azuara Garcia, Efreen Palacios Rosas, Alfonso Garcia-Ferrrer and Pilar Montesano's Barrios (2017),"Multi-objective spatial optimization: Sustainable Land Use Allocation at Sub- Regional Scale", Sustainability, June 2017, MAPT.

[4] R. Vinnet, C. Fonteix & I. Marc (2007),"Multi criteria optimization using a genetic algorithm for determine a pareto set", International Journal of Systems Science, pages 255-260, Volume 27, Issue-2.

[5] Nyoman Gunantara (2018),"A review of multi-objective optimization: Methods and its Applications", Cogent Engineering, Volume5, 2018 -Issue1.

[6] Banker Rd, Charnes. A Cooper. W.W and Schinner, A.P (1981), "A Bi- Extremal Principle for Frontier Estimation and Efficiency Evaluation". Management Science, 27, 1370-1382.

[7] Cak Lovell and APB Rouse (2003), "Equivalent standard DEA models to provide super efficiency Scores", Journal of Operational Research Society, Vol. 54, and Pp: 101-108.

[8] Joe Zhu (2003). "Imperic Data Envelopment Analysis (Idea): A review And Improvement with an Application ".European Journal of Operation Research 513-529.

[9] Alien R, Thanassoulis E. (2004),"Improving Envelopment in Data Envelopment Analysis", European Journal of Operations Research 363-379.

[10] Ray R.C And Kim H.J (1995) "Cost Efficiency In The US Steel Industry: A Non –Parametric Analysis Using Data Envelopment Analysis", European Journal of Operation Research 80,654-671.

[11] Sengupta J.K and Sfeir R.E (1988) "Efficiency Measurement by Data Envelopment Analysis with Econometric Application', Applied Economics 20,285-294.

[12] Sengupta J.K. (2000), "Stochastic and dynamic efficiency analysis: Economics of Data Envelopment Analysis, World Scientific London.

[13] Yao Chen, (2005), "Measuring Super-Efficiency in DEA In The Presence of Infeasibility", European Journal of Operation Research, 545-551.

[14] B. Venkateswarlu, B. Mahaboob , J. Ravi sankar and B. Madhusudhana Rao (2018),"An Application of Goal Programming in Data Envelopment Analysis", International Journal of Engineering & Technology, Vol.(7), issue 4, Pp: 523-525.

[15] B Venkateswarlu, M Mubashir Unnissa and B Mahaboob (2016), "Estimating Cost Analysis using Goal Programming", Indian Journal of Science and Technology, Vol. (9), Issue (4).

[16] B Venkateswarlu, B Mahaboob, C Subbarami Reddy and B Madhusudhana Rao (2017), "Fitting of full Cobb-Douglas and full VRTS cost frontiers by solving goal programming problem", Materials Science and Engineering Conference Series, Vol. (263), Issue (4).

[17] B. Mahaboob, B. Venkateswarlu, K.A. Azmath, C. Narayana, J. Peter Praveen (2019), "On OLS Estimation of Stochastic Linear Regression Model", International Journal of Engineering and Advanced Technology (IJEAT), ISSN: 2249 – 8958, Volume-8 Issue-6, Pp: 1953-1955, Volume-8 Issue-6.

[18] B. Venkateswarlu, B. Mahaboob, K.A. Azmath, C. Narayana (2019), "An Application of Linear Programming in the Estimation of Technical Efficiency of DMU", International Journal of Engineering and Advanced Technology, (IJEAT), and ISSN: 2249 – 8958.

[19] B. Venkateswarlu, B. Mahaboob, C. Narayana, C. Subbarami Reddy (2019), "Evaluation of Slack Based Efficiency of a Decision Making Unit", International journal of scientific & Technology research.

[20] B. Venkateswarlu, B. Mahaboob, C. Narayana, C. Subbarami Reddy (2019), New Results In Production Theory By Applying Goal Programming,, International journal of scientific & Technology research., Vo.(8), Issue (12).