Poisson Semigroups With Inverse And Biinverse Square Potentials

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Abstract: In this paper, by means of the principle of subordination, we give explicit formulas for the Poisson kernels with inverse and biinverse square potentials using the well known formulas for the corresponding heat kernels.

Index Terms: Poisson semigroup, inverse square potential, biinverse square potential, Gauss hypergeometric function, Appel hypergeometric functions.

1 INTRODUCTION

The classical Poisson problem on R^n is given by

$$\begin{cases} \frac{\partial^2 u(y,x)}{\partial y^2} = -\Delta u(y,x), x \in R^n, y > 0 \\ u(0,x) = u_0(x) \end{cases} \quad (1.1)$$

where $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the classical Laplacian on R^n.

The solution of this problem can be described via the Poisson kernel, where if the functions u_0 belong to the Lebesgue space $L^2(R^n, dx)$ the solution of this problems is given by(Folland [5], p. 93)

$$u(y,x) = \frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}} \int_{R^n} \frac{y}{(y^2+|x-x'|^2)^{(n+1)/2}} u_0(x') dx'. \quad (1.2)$$

In this paper we solve explicitly the following Poisson Problems with inverse and bi-inverse square potentials on IR_+ and IR_+^2

$$\begin{cases} \Delta_\nu u(z,x) + \frac{\partial^2}{\partial z^2} u(z,x) = 0 & (z,x) \in R_+ \times R_+, \\ u(0,x) = u_0(x) & u_0 \in C_0^\infty(R_+) \end{cases} \quad (1.3)$$

and

$$\begin{cases} \Delta_{\nu,\mu} u(z,p) + \frac{\partial^2}{\partial z^2} u(z,p) = 0 & (z,p) \in R_+ \times R_+^2, \\ u(0,p) = u_0(p) & u_0 \in C_0^\infty(R_+^2) \end{cases} \quad (1.4)$$

where

$$\Delta_\nu = \frac{\partial^2}{\partial x^2} + \frac{1/4-\nu^2}{x^2}, \quad (1.5)$$

and

$$\Delta_{\nu,\mu} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1/4-\nu^2}{x^2} + \frac{1/4-\mu^2}{y^2}, \quad (1.6)$$

are the Schrödinger operators with inverse and biinverse square potentials.

2 POISSON EQUATION WITH INVERSE SQUARE POTENTIAL

It seems that the first formula for the Poisson kernel with inverse square potential is obtained in (Muckenhoupt-Stein [9]):

$$P^{\nu}(z,x,x') = \frac{(1+2\nu)}{\pi} (x\,x')^{\nu+\frac{1}{2}} z \int_0^\pi \frac{\sin^{2\nu} y\, dy}{\left(y^2+x^2+x'^2-z^2\right)^{\nu+\frac{3}{2}}}.$$

An other formula is given in (Betancor et al. [1]),

$$P^\lambda(z,x,x') = C \frac{(4xx')^{1-\lambda}}{((x+y)^2+z^2)((x-y)^2+z^2)} \times F\left(\frac{\lambda+1}{2}, \frac{\lambda+2}{2}, \frac{1+2\lambda}{2}, \left(\frac{2xx'}{z^2+x^2+x'^2}\right)^2\right),$$

where

$$C = \frac{2\Gamma(\lambda+1)}{\sqrt{\pi}\Gamma(\lambda+1/2)}.$$

In this section we give an other formula for the Poisson kernel with inverse square potential, for this we start by the following lemma.

Lemma 2.1 For $\nu > -1/2$, the following formula holds

$$\int_0^\infty e^{-px} x^{\alpha-1} I_\nu(cx) dx = \frac{\Gamma(\alpha+\nu)\Gamma(\nu+1/2)}{\sqrt{\pi}\Gamma(2\nu+1)} \frac{(2c)^\nu}{(p-c)^{\alpha+\nu}} \times {}_2F_1(\alpha+\nu, \nu+\tfrac{1}{2}, 2\nu+1, -\tfrac{2c}{p-c}), \quad (2.1)$$

where the Gauss hypergeometric function is defined by:

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad |z| < 1,$$

and as usual $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ is the Pochhamer symbol, Γ is the classical Euler function.

Proof. Replacing by the formula (Temme [16], p. 237),

$$I_\nu(z) = \frac{(2z)^\nu e^z}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^1 e^{-2zt} [t(1-t)]^{\nu-1/2} dt, \quad (2.2)$$

in the left hand side of (2.1), changing the order of integration by Fubini theorem and the integral representation of the Gauss hyprgeometric function (Magnus [8], p. 54),

$${}_2F_1(\alpha,\beta,\gamma,z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 u^{\beta-1}(1-u)^{\gamma-\beta-1}(1-uz)^{-\alpha} du, \quad (2.3)$$

with $Re\gamma > Re\beta > 0$ and $\arg(1-z) < \pi$, we arrive at the result of the lemma.

Theorem 2.1 For $\nu > -1/2$, the Poisson kernel with inverse square potential on the half real line IR^+ is given by

$$P^\nu(z,x,x') = \frac{2\Gamma(\nu+3/2)\Gamma(\nu+1/2)}{\pi\Gamma(2\nu+1)} \frac{(4xx')^\nu z}{(z^2+(x-x')^2)^{\nu+3/2}} \times F\left(\nu+3/2, \nu+1/2, 2\nu+1, \frac{-4xx'}{z^2+(x-x')^2}\right).$$

(2.4)

Proof. We recall the formula (Strichartz [15], p. 50).

$$\frac{e^{-ty}}{t} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-ut^2} u^{-1/2} e^{-y^2/4u} du. \quad (2.5)$$

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Set $t = z$ and $y = \sqrt{-\Delta_\nu}$ we obtain

$$e^{-z\sqrt{-\Delta_\nu}} = \frac{z}{\sqrt{\pi}}\int_0^\infty e^{-uz^2}u^{-1/2}e^{\Delta_\nu/4u}du.$$

Where $e^{t\Delta_\nu}$ is the heat kernel with the inverse square potential given in (Calin et al. [3], p. 68)

$$e^{t\Delta_\nu}(t,x,x') = \frac{(xx')^{1/2}}{2t}e^{\frac{-(x^2+x'^2)}{4t}}I_\nu(\frac{xx'}{2t}).$$

$$e^{-z\sqrt{-\Delta_\nu}} = \frac{2z}{\sqrt{\pi}}\int_0^\infty u^{1/2}e^{-(y^2+x^2+x'^2)u}I_\nu(2xx'u)du.$$

We use the Lemma 2.1, and the proof of the theorem is Finished.

3 POISSON EQUATION WITH BIINVERSE SQUARE POTENTIAL

The Schrodinger operator with biinverse square potential is considered in (Boyer [2]), for the heat and the wave equation with biinverse square potential see (Ould Moustapha [11], [12]). We recall the following formula for the heat kernel on the the quarter plane IR_+^2.

Proposition 3.1 (Ould Moustapha [11], [12]) The Schwartz integral kernel of the heat operator with biinverse square potential $e^{t\Delta_{\nu,\mu}}$ on the quarter plane IR_+^2 is given, for $p = (x,y)$, $p' = (x',y') \in IR_+^2$ and $t \in IR_+$, by

$$H_{\nu,\mu}^+(t,p,p') = \frac{(xx')^{\frac{1}{2}}(yy')^{\frac{1}{2}}}{4t^2}e^{-(|p|^2+\frac{|p'|^2)}{4t}}I_\nu\left(\frac{xx'}{2t}\right)I_\mu\left(\frac{yy'}{2t}\right). \quad (3.1)$$

Lemma 3.1 For $Re\nu > -1/2$, $Re\mu > -1/2$ and $2|c| + 2|c'| < |p - c - c'|$, the following formula holds

$$\int_0^\infty e^{-px}x^{\alpha-1}I_\nu(cx)I_\mu(c'x)dx = \frac{\Gamma(\alpha+\nu+\mu)\Gamma(\nu+1/2)\Gamma(\mu+1/2)}{\pi\Gamma(2\nu+1)\Gamma(2\mu+1)}\frac{(2c)^\nu(2c')^\mu}{(p-c-c')^{\alpha+\nu+\mu}}\times F_2\left(\begin{smallmatrix}\alpha+\nu+\mu,\nu+1/2,\mu+1/2\\ 2\nu+1,2\mu+1\end{smallmatrix}\Big| -\frac{2c}{p-c-c'}, -\frac{2c'}{p-c-c'}\right), \quad (3.2)$$

where F_2 is the two variables Appell hypergeometric function: (see Erdélyi et al. [4] p. 224)

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',z,z') = \sum_{m=0}^\infty\sum_{n=0}^\infty \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_m(\gamma')_n m!n!}z^m z'^n, \quad |z|+|z'| < 1. \quad (3.3)$$

Proof. This lemma is a consequence of the formula (2.2), Fubini theorem and the integral representation of the Appel hyprgeometric function of two variables (Erdélyi et al. [4], p.230) for $\Re\beta > 0$, $\Re e\beta' > 0$, $\Re e(\gamma - \beta) > 0$ and $\Re e(\gamma' - \beta') > 0$:

$$F_2(\alpha,\beta,\beta',\gamma,\gamma',z,z') = c\int_0^1\int_0^1 (1-u)^{\gamma-\beta-1}\times (1-v)^{\gamma'-\beta'-1}u^{\beta-1}v^{\beta'-1}(1-uz-vz')^{-\alpha}dudv, \quad (3.4)$$

where

$$c = \frac{\Gamma(\gamma)\Gamma(\gamma')}{\Gamma(\beta)\Gamma(\beta')\Gamma(\gamma-\beta)\Gamma(\gamma'-\beta')}.$$

Theorem 3.1 For $\nu > -1/2$ and $\mu > -1/2$, the Poisson problem (1.4) with biinverse square potential on the quarter plane R_+^2 has the unique solution given by

$$u(z,p) = \int_{R_+^2} P_{\nu,\mu}(z,p,p')u_0(p')dp',$$

where

$$P_{\nu,\mu}(z,p,p') = \frac{4\Gamma(\alpha)\Gamma(\beta_1)\Gamma(\beta_2)}{\pi^{3/2}\Gamma(2\beta_1)\Gamma(2\beta_2)}\frac{(4xx')^\nu(4yy')^\mu z}{(z^2+|p-p'|^2)^\alpha}\times F_2\left(\begin{smallmatrix}\alpha,\beta_1,\beta_2\\ 2\beta_1,2\beta_2\end{smallmatrix}\Big|\frac{-4xx'}{z^2+|p-p'|^2}, \frac{-4yy'}{z^2+|p-p'|^2}\right), \quad (3.5)$$

F_2 is the two variables Appel hypergeometric function given by (3.3), $\alpha = \nu + \mu + 5/2$, $\beta_1 = \nu + 1/2$ and $\beta_2 = \mu + 1/2$.

Proof. By setting $t = z$ and $y = \sqrt{-\Delta_{\nu,\mu}}$ in the formula (2.5) we can write

$$e^{-z\sqrt{-\Delta_{\nu,\mu}}} = \frac{z}{2\sqrt{\pi}}\int_0^\infty e^{-uz^2}u^{-1/2}e^{\Delta_{\nu,\mu}/4u}du \quad (3.6)$$

$$e^{-z\sqrt{-\Delta_{\nu,\mu}}} = \frac{4z}{\sqrt{\pi}}(xyx'y')^{1/2}\int_0^\infty e^{-u(z^2+|p|^2+|p'|^2)}u^{3/2}I_\nu(2xx'u)I_\mu(2yy'u)du. \quad (3.7)$$

Using the Lemma 3.1 we have the result of the theorem. For the values of parameters $\mu = \nu - 3/2$, we can also, express the Poisson kernel (3.5) in terms of the Appel hypergeometric function F_1.

$$F_1(a,b,b',c,w,z) = \sum_{n,m=0}^\infty \frac{(a)_{m+n}(b)_m(b')_n}{(c)_{m+n}m!n!}w^m z^n, \quad |w|,|z| < 1. \quad (3.8)$$

Proposition 3.2 For $\nu > -1/2$,

$$P_{\nu,\nu-3/2}^+(z,p,p') = C_\nu(xx')^\nu(yy')^{\nu-3/2}\frac{a_2^{\nu+1}}{a_1^{3\nu+3/2}}z\times F_1\left(\nu-1, 1/2+\nu, 1/2+\nu, 2\nu-2, \frac{4(x'x')}{a_2}, \frac{4(yy')}{a_1}\right).$$

where $a_1 = z^2 + |p - p'|^2$ and $a_2 = z^2 + |p + p'|^2$.

Proof. This proposition is a consequence of the formulas (3.5) the following formula [14].

$$F_2(d,a,a';d,c',x,y) = (1-x)^{-a}F_1(a',a,d-a,c',y/(1-x),y)$$

to obtain

$$F_2\left(\alpha,\beta,\beta',2\beta,2\beta',\frac{4xx'}{z^2+|p-p'|^2},\frac{4yy'}{z^2+|p-p'|^2}\right) = \left(\frac{a_1}{a_2}\right)^{\nu+1/2}\times F_1\left(\nu-1, 1/2+\nu, 1/2+\nu, 2\nu-2, \frac{4yy'}{a_1}, \frac{4yy'}{a_2}\right)$$

and we replace in the formula (3.5).

We can give some other particular cases in terms of the Gauss Hypergeometric function using the following particular cases of F_2 (Saad-Hall [10]),

$$F_2(d,0,a';c,c',x,y) = {}_2F_1(d,a',c',y),$$
$$F_2(d,a,0,c,c',x,y) = {}_2F_1(d,a,c,x),$$
$$F_2(d;a,a',a,a';x,y) = (1-x-y)^{-d},$$
$$F_2(d;a,a',d,a';x,y) = (1-y)^{a-d}(1-x-y)^{-a},$$
$$F_2(d;a,a',d,a;x,y) = (1-x)^{a'-d}(1-x-y)^{-a'},$$

See also (Gradshteyn-Ryzhik [7] p.1019 formula 9.182)

$$F_2(d;a,a',a,c';x,y) = (1-x)^{-d}{}_2F_1\left(d,a',c',\frac{y}{1-x}\right),$$

by analogy we have

$$F_2(d;a,a',c,a';x,y) = (1-y)^{-d}{}_2F_1\left(d,a,c,\frac{x}{1-y}\right).$$

4 APPLICATION TO BESSEL OPERATOR AND SCHRÖDINGER EQUATION IN NON-INTEGER DIMENSIONS

Note that the Schrödinger operator with inverse square potential Δ_ν is related to the Bessel operator

$$\widetilde{\Delta}_\nu = \frac{\partial^2}{\partial x^2} + \frac{1-2\nu}{x}\frac{\partial}{\partial x}, \quad (4.1)$$

by

$$x^{\nu-1/2}\Delta_\nu x^{-\nu+1/2} = \widetilde{\Delta}_\nu, \quad (4.2)$$

$$\widetilde{\Delta}_\nu = \frac{\partial^2}{\partial x^2} + \frac{1-2\nu}{x}\frac{\partial}{\partial x}. \quad (4.3)$$

The Schrödinger operator with biinverse square potential $\Delta_{\nu,\mu}$, is related to the Laplacian in noninteger dimension given in (Frank-Stillinger[6])and(Palmer-Stavrinou [13])by

$$\widetilde{\Delta}_{\nu,\mu} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1-2\nu}{x}\frac{\partial}{\partial x} + \frac{1-2\mu}{y}\frac{\partial}{\partial y}, \quad (4.4)$$

via the formula

$$x^{\nu-1/2}y^{\mu-1/2}\Delta_{\nu,\mu}x^{-\nu+1/2}y^{-\mu+1/2} = \widetilde{\Delta}_{\nu,\mu}. \quad (4.5)$$

Corollary 4.1 For ν real, if $\widetilde{H}^\nu(t,x,x')$ and $\widetilde{P}^\nu(y,x,x')$ are respectively the heat and the Poisson kernel associated to the

Bessel operator $\widetilde{\Delta}_\nu$, then we have

$$\widetilde{H}_\nu(t,x,x') = (xx')^{1/2-\nu} H_\nu(t,x,x').$$
$$\widetilde{P}_\nu(y,x,x') = (xx')^{1/2-\nu} P_\nu(y,x,x').$$

Where $H_\nu(t,x,x')$ and $P_\nu(y,x,x')$ are given by (2.7) and (2.4).

We have the following results for the Laplacian in noninteger dimension

Corollary 4.2 Let $\widetilde{H}^{\nu,\mu}(t,p,p')$ and $\tilde{P}^{\nu,\mu}(t,p,p')$ be the heat and the Poisson kernel associated to the Laplacian in noninteger dimension $\tilde{\Delta}_{\nu,\mu}$, Then for ν,μ real, we have ,

$$\widetilde{H}_{\nu,\mu}(t,p,p') = (xx')^{1-2\nu}(yy')^{\frac{1}{2}-\mu} H_{\nu,\mu}(t,p,p'),$$
$$\widetilde{P}_{\nu,\mu}(z,p,p') = (xx')^{1-2\nu}(yy')^{1/2-\mu} P_{\nu,\mu}(z,p,p'),$$

Where $H_{\nu,\mu}(t,p,p')$ and $P_{\nu,\mu}(z,p,p')$ are given by (3.1) and (3.5).

Proof. The proof of this corollary is simple and is left to the reader.

5 NUMERICAL EXAMPLES

The Mathematica plotting gives a much better idea of what happens for $p' = (1,1)$ (see figures 1, 2, 3and 4 below).

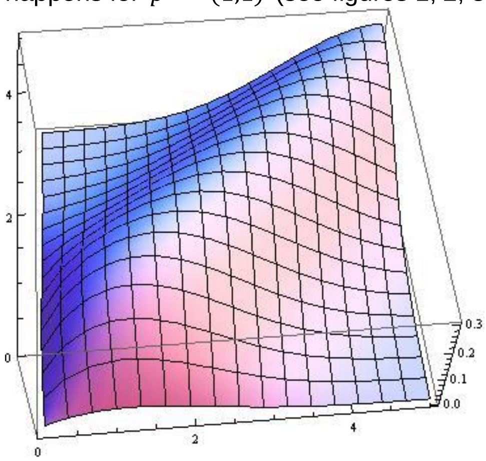

Figure 1: *Graphic for the heat kernel on the half real line* IR_+, $t = 1, \nu = 1/5$.

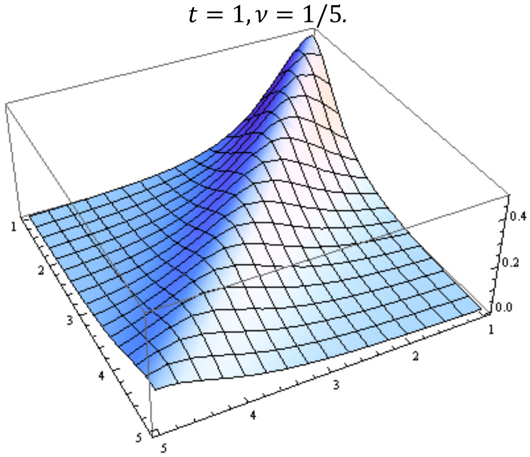

Figure 2: *Graphic for the Poisson kernel on the half real line* IR^+, $y = 1, \nu = 1$

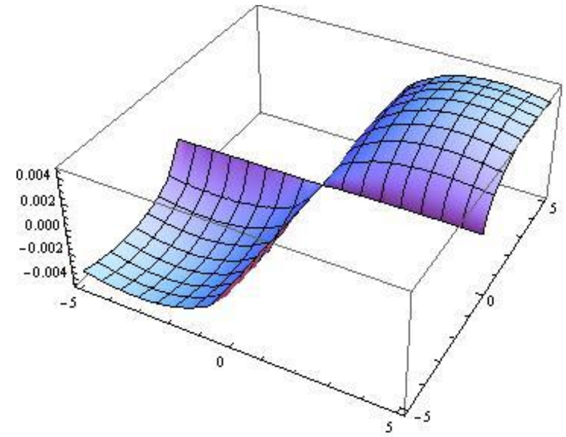

Figure 3: *Graphic for the heat kernel on the quarter* IR_+^2, $t = 1, \nu = 1/5$.

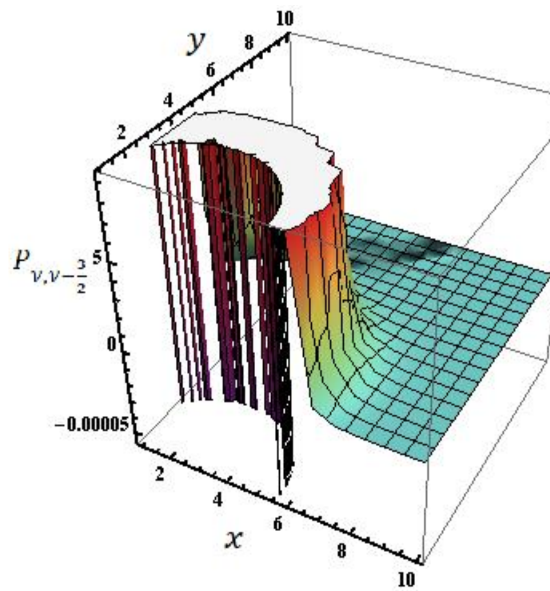

Figure 4: *Graphic for the Poisson kernel on the half real line* IR^+, $y = 1, \nu = 3, \mu = 3/2$.

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