Resolvent Kernels Of Dirac, Euler Operators And Harmonic Oscillators

Anouar Saidi, Mohamed Vall Ould Moustapha

Abstract: In this article, we give a new method based on the Bargmann transform to compute the resolvent kernel and the eigenvectors of generalized Dirac, Euler operators and harmonic oscillators.

Index Terms: Bargmann transform, harmonic oscillator, integral transform, intertwinning operator, resolvent kernel, Green's function, eigenvectors.

1 INTRODUCTION

Green's functions were so-called after the famous mathematician George Green, who developed this notion in the nineteenth century [1],[2],[3]. Green's functions method enable the solution of a non-homogeneous differential equation to be related to an integral operator. This method has proved very useful in solving both partial and exact differential equations ([4], [5]).

Let L be a differential operator on $\mathbb{K}$ and $\lambda \in \mathbb{K}$ where $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. A function $G: \mathbb{K}^3 \longrightarrow \mathbb{K}$ (considered here as a distribution) is said to be Green's function of L if

$$(L+\lambda^2)G(x,x',\lambda)=\delta(x-x'),$$

where $\delta(x-x')$ is the Dirac distribution defined by

$$\langle \delta(x-x'), \varphi(x)\rangle = \varphi(x')$$

This gives us the solution of the following differential equation

$$(L+\lambda^2)\, u(x) = f(x). \qquad (1.1)$$

where f is a fixed smooth function. In fact, if G is the Green's function associated to the operator L, then

$$(L+\lambda^2)\left(\int_K G(x,x',\lambda)f(x')dx'\right)$$
$$=\left(\int_{\mathbb{K}} (L+\lambda^2)G(x,x',\lambda)f(x')dx'\right)$$
$$=\int_{\mathbb{K}} \delta(x-x')f(x')\,dx' = f.$$

So the solution of (1.1) is

$$u(x)=\int_K G(x,x',\lambda)f(x')\,dx'.$$

The function G is also called the resolvent kernel of the differential operator L. Our aim in this work is to give a new method to compute explicitly the resolvent kernels and the eigenvectors associated to some real and complex operators of type Dirac, Euler and harmonic oscillator. Our method is based on a parametrized form of the well-known Bargmann transform [6], [7] given in [8] as follows

$$[B_a f](z) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{-\infty}^{\infty} f(x)\, e^{2axz-ax^2-\frac{a\,z^2}{2}}\, dx \qquad (1.2)$$

For $a>0$, we define a Gaussian measure on $\mathbb{C}$ as follows

$$d\lambda_a(z)=\frac{a}{\pi}e^{-a|z|^2}dz.$$

The mapping B_a is an isometry from $L^2(\mathbb{R})$ which is the subspace of all entire functions in $L^2(\mathbb{C}, d\lambda_a)$ often called the Fock space and denoted F_a^2.

Its inverse is given by

$$[B_a^{-1}f](x) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}} \int_{\mathbb{C}} f(z)\, e^{2ax\bar{z}-ax^2-\frac{a}{2}\bar{z}^2}\, d\lambda_a(z) \qquad (1.3)$$

The Bargmann transform has been widely studied in mathematics see for example [9], [10], [11], [12], [13], [14], [15]. We use it here as an intertwining operator that transports the harmonic oscillator to an Euler operator.

This paper is outlined as follows:

In section 2, we compute explicitly the resolvent kernels for the real generalized Dirac operator d_x^a, the real Euler operator e_x^a and the complex harmonic oscillator H_z^a respectively defined by

$$d_x^a = \frac{\partial}{\partial x} + ax \qquad (1.4)$$
$$e_x^a = ax\frac{\partial}{\partial x} + \frac{a}{2} \qquad (1.5)$$
$$H_z^a = \frac{\partial^2}{\partial z^2} - \frac{a^2z^2}{4} \qquad (1.6)$$

In section 3, we calculate the eigenvectors of the complex Dirac operator D_z^a, the complex Euler operator E_z^a and the real harmonic oscillator h_x^a which are respectively defined by

$$D_z^a = \frac{\partial}{\partial z} \qquad (1.7)$$
$$E_z^a = -2az\frac{\partial}{\partial z} - a \qquad (1.8)$$
$$h_x^a = \frac{\partial^2}{\partial x^2} - a^2x^2 \qquad (1.9)$$

2 RESOLVENT KERNELS

Our purpose in this paragraph is to give the resolvent kernel of the complex harmonic oscillator defined in (1.6). For this aim, we compute the resolvent kernel for generalized real Dirac operator and real Euler operators defined respectively in (1.4) and (1.5).

Lemma 2.1 The resolvent kernel of the real Dirac operator $d_x = \frac{\partial}{\partial x}$ is given by

$$G(x,x',\lambda) = \begin{cases} e^{-\lambda^2(x-x')} & if \quad x \geq x', \\ 0 & if \quad x < x'. \end{cases}$$

Proof. We have

- *Anouar Saidi, PHD, College of Sciences and Arts of Gurayat, Jouf University, Kingdom of Saudi Arabia. Faculty of Sciences of Monastir, 5019 Monastir-Tunisia. E-mail address: saidi.anouar@yahoo.fr,*
- *Mohamed Vall Ould Moustapha, PHD, College of Sciences and Arts of Gurayat, Jouf University, Kingdom of Saudi Arabia. Faculté des Sciences et Techniques, Université de Nouakchott Al- Aasriya, Nouakchott-Mauritanie E-mail address: mohamedvall.ouldmoustapha230@gmail.com*

$$\left\langle\left(\frac{\partial}{\partial x}+\lambda^2\right)G(x,x',\lambda),\varphi(x)\right\rangle=\left\langle G(x,x',\lambda),\left(-\frac{\partial}{\partial x}+\lambda^2\right)\varphi(x)\right\rangle$$
$$=-\int_{x'}^{\infty}e^{-\lambda^2(x-x')}\varphi'(x)\,dx+\int_{x'}^{\infty}\lambda^2e^{-\lambda^2(x-x')}\varphi(x)\,dx.$$

An integration by parts of the first integral gives

$$\left\langle\left(\frac{\partial}{\partial x}+\lambda^2\right)G(x,x',\lambda),\varphi(x)\right\rangle=\varphi(x').$$

Thus we obtain

$$\left(\frac{\partial}{\partial x}+\lambda^2\right)G(x,x',\lambda)=\delta(x-x').\square$$

Theorem 2.1 The resolvent kernel of the generalized real Dirac operator $d_x^a=\frac{\partial}{\partial x}+ax$ is given by

$$G(x,x',\lambda)=\begin{cases}e^{-\lambda^2(x-x')}e^{-\frac{a}{2}x^2}e^{\frac{a}{2}x'^2} & if \quad x\geq x',\\ 0 & if \quad x<x'.\end{cases}$$

Proof. We have

$$\left\langle\left(\frac{\partial}{\partial x}+ax+\lambda^2\right)G(x,x',\lambda),\varphi(x)\right\rangle=$$
$$\left\langle G(x,x',\lambda),\left(-\frac{\partial}{\partial x}+ax+\lambda^2\right)\varphi(x)\right\rangle=$$
$$\int_{\mathbb{R}}-\varphi'(x)G(x,x',\lambda)\,dx+\int_{\mathbb{R}}(\lambda^2+ax)\varphi(x)G(x,x',\lambda)\,dx=$$
$$-\int_{x'}^{\infty}e^{-\lambda^2(x-x')}\;e^{-\frac{a}{2}x^2}\;e^{\frac{a}{2}x'^2}\varphi'(x)dx$$
$$+\int_{x'}^{\infty}(\lambda^2+ax)\,e^{-\lambda^2(x-x')}e^{-\frac{a}{2}x^2}e^{\frac{a}{2}x'^2}\varphi(x)\,dx.$$

An integration by parts of the first integral gives

$$\left\langle\left(\frac{\partial}{\partial x}+ax+\lambda^2\right)G(x,x',\lambda),\varphi(x)\right\rangle=\varphi(x').$$

Thus we obtain

$$\left(\frac{\partial}{\partial x}+ax+\lambda^2\right)G(x,x',\lambda)=\delta(x-x').\square$$

Theorem 2.2 The resolvent kernel of the real generalized Euler operator $e_x^a=ax\frac{\partial}{\partial x}+\frac{a}{2}$ is given by

$$G(x,x',\lambda)=\begin{cases}\frac{1}{a}\,x^{\left(-\frac{\lambda^2}{a}-\frac{1}{2}\right)}x'^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)} & if \quad x\geq x',\\ 0 & if \quad x<x'.\end{cases}$$

Proof. We have

$$\left\langle\left(ax\frac{\partial}{\partial x}+\frac{a}{2}+\lambda^2\right)\;G(x,x',\lambda),\varphi(x)\right\rangle$$
$$=\left\langle G(x,x',\lambda),\left(-\frac{\partial}{\partial x}ax+\frac{a}{2}+\lambda^2\right)\;\varphi(x)\right\rangle$$
$$=\left\langle G(x,x',\lambda),\left(-ax\frac{\partial}{\partial x}-\frac{a}{2}+\lambda^2\right)\varphi(x)\right\rangle$$
$$=\lim_{\varepsilon\to 0}\int_{|x-x'|>\varepsilon}G(x,x',\lambda)\left(-ax\varphi'(x)+\left(-\frac{a}{2}+\lambda^2\right)\varphi(x)\right)dx$$
$$=\lim_{\varepsilon\to 0}\int_{|x-x'|>\varepsilon}G(x,x',\lambda)(-ax\varphi'(x))\,dx$$
$$+\lim_{\varepsilon\to 0}\int_{|x-x'|>\varepsilon}G(x,x',\lambda)\left(-\frac{a}{2}+\lambda^2\right)\varphi(x)\,dx$$
$$=\frac{1}{a}\lim_{\varepsilon\to 0}\left(-\int_{x'+\varepsilon}^{\infty}a\,x^{-\frac{\lambda^2}{a}+\frac{1}{2}}x'^{\frac{\lambda^2}{a}-\frac{1}{2}}\varphi'(x)\,dx\right.$$
$$\left.+\int_{x'+\varepsilon}^{\infty}\left(-\frac{a}{2}+\lambda^2\right)\;x^{-\frac{\lambda^2}{a}-\frac{1}{2}}x'^{\frac{\lambda^2}{a}-\frac{1}{2}}\varphi(x)\,dx\right).$$

An integration by parts of the first integral gives

$$\left\langle\left(ax\frac{\partial}{\partial x}+\frac{a}{2}+\lambda^2\right)G(x,x',\lambda),\varphi(x)\right\rangle=\varphi(x').\square$$

In what follows, we give some formulas involving Bargmann transform.

Lemma 2.2 ([8]) For $a>0$, we have

1. $\left[B_{\frac{a}{2}}(xf)\right](z)=\left(\frac{1}{a}\frac{\partial}{\partial z}+\frac{z}{2}\right)\left[B_{\frac{a}{2}}f\right](z)$.
2. $\left[B_{\frac{a}{2}}\left(\frac{\partial}{\partial x}f\right)\right](z)=\left(\frac{\partial}{\partial z}-\frac{a}{2}z\right)\left[B_{\frac{a}{2}}f\right](z)$.
3. $\left[B_{\frac{a}{2}}\left(\left(\frac{\partial}{\partial x}-ax\right)\;f\right)\right](z)=-az\left[B_{\frac{a}{2}}f\right](z)$.
4. $\left[B_{\frac{a}{2}}\left(\left(\frac{\partial}{\partial x}+ax\right)\;f\right)\right](z)=2\frac{\partial}{\partial z}\left[B_{\frac{a}{2}}f\right](z)$.

The following proposition is a direct consequence of lemma 2.2.

Proposition 2.1 Let e_x^a and H_z^a be respectively the real Euler operators and the complex harmonic oscillator given in (1.5) and (1.6). Then, we obtain

$$\left[B_{\frac{a}{2}}(e_x^af)\right](z)=H_z^a\left[B_{\frac{a}{2}}f\right](z).$$

Lemma 2.3 Let a and c be real positive numbers then the following lemma hold

$$\left[B_{\frac{a}{2}}\left(e^{-cx^2}\right)\right](z)=\left(\frac{a}{\pi}\right)^{\frac{1}{4}}\sqrt{\frac{\pi}{c+\frac{a}{2}}}\;e^{\frac{a}{4}z^2\left(\frac{a-2c}{a+2c}\right)}.$$

Proof. For $c>0$ we can write

$$\left[B_{\frac{a}{2}}\left(e^{-cx^2}\right)\right](z)=\left(\frac{a}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}e^{-cx^2}\;e^{axz-\frac{a}{2}x^2-\frac{a}{4}z^2}\,dx=$$
$$\left(\frac{a}{\pi}\right)^{\frac{1}{4}}\int_{-\infty}^{\infty}e^{-\left(c+\frac{a}{2}\right)\left(x-\frac{a}{a+2c}z\right)^2+\left(c+\frac{a}{2}\right)\frac{a^2}{(a+2c)^2}z^2-\frac{a}{4}z^2}dx$$

Then, $\left[B_{\frac{a}{2}}\left(e^{-cx^2}\right)\right](z)=$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{4}}e^{\left(\frac{a^2}{2(a+2c)}-\frac{a}{4}\right)z^2}\int_{-\infty}^{\infty}e^{-\left(c+\frac{a}{2}\right)\left(x-\frac{a}{a+2c}z\right)^2}dx$$

Now lemma 2 of [10] implies that

$$\int_{-\infty}^{\infty}e^{-\left(c+\frac{a}{2}\right)\left(x-\frac{a}{a+2c}z\right)^2}dx=\sqrt{\frac{\pi}{c+\frac{a}{2}}}$$

Thus, we obtain

$$\left[B_{\frac{a}{2}}\left(e^{-cx^2}\right)\right](z)=\left(\frac{a}{\pi}\right)^{\frac{1}{4}}\sqrt{\frac{\pi}{c+\frac{a}{2}}}\;e^{\frac{a}{4}z^2\left(\frac{a-2c}{a+2c}\right)}.\;\square$$

Theorem 2.3 The resolvent kernel of the complex harmonic oscillator H_z^a is given by

$$G(z,z',\lambda)=\frac{e^{-\frac{a}{2}|z'|^2}}{2\sqrt{2}\pi}\int_{\mathbb{R}}K(v,z,z',\lambda)dv.$$

where $K(v,z,z',\lambda)=$

$$sgn(v-1)\frac{v^{\frac{\lambda^2}{a}-\frac{1}{2}}}{\sqrt{1+v^2}}e^{\frac{a}{4}\left(\frac{1-v^2}{1+v^2}\right)z^2}\;e^{\frac{a}{4}\left(\frac{v^2-1}{1+v^2}\right)\overline{z'}^2}\;e^{a\left(\frac{v}{1+v^2}\right)z\overline{z'}}\quad(2.1)$$

Proof. Let u be the solution of $(H_z^a+\lambda^2)u=f$. Then, by applying the inverse Bargmann transform to this equation and using proposition 2.1, we obtain

$(e_x^a+\lambda^2)[B_{\frac{a}{2}}^{-1}u](x)=[B_{\frac{a}{2}}^{-1}f](x)$.

Thus, by theorem 2.2, we obtain

$$\left[B_{\frac{a}{2}}^{-1}u\right](x)=\frac{1}{a}\int_{-\infty}^{x}x^{\left(-\frac{\lambda^2}{a}-\frac{1}{2}\right)}x'^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}[B_{\frac{a}{2}}^{-1}f](x')\,dx'$$

where

$$[B_{\frac{a}{2}}^{-1}f](x')=\left(\frac{a}{\pi}\right)^{\frac{1}{4}}\int_{\mathbb{C}}f(z')e^{a\,x'\overline{z'}-\frac{a}{2}x'^2-\frac{a\,\overline{z'}^2}{4}}\,d\lambda_{\frac{a}{2}}(z').$$

Thus

$$\left[B_{\frac{a}{2}}^{-1}u\right](x)=\int_{\mathbb{R}}H(x-x')\frac{x^{\left(-\frac{\lambda^2}{a}-\frac{1}{2}\right)}\;x'^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{a}[B_{\frac{a}{2}}^{-1}f](x')dx',$$

where H is the Heaviside function.

Since $u(z) = [B_{\frac{a}{2}}[B_{\frac{a}{2}}^{-1}u]](z)$, then

$$u(z) = \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}}\int_{\mathbb{R}} H(x-x')\frac{x^{\left(-\frac{\lambda^2}{a}-\frac{1}{2}\right)}\; x'^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{2a}\gamma(x,x',z)dx'dx$$

where $\gamma(x,x',z) =$

$$\int_{\mathbb{C}} f(z')\, e^{ax'\overline{z'}-\frac{a}{2}x'^2-\frac{a\overline{z'}^2}{4}}\, e^{axz-\frac{a}{2}x^2-\frac{a z^2}{4}}\, d\lambda_{\frac{a}{2}}(z')$$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} f(z') \int_{\mathbb{R}}\int_{\mathbb{R}} H(x-x')\frac{x^{\left(-\frac{\lambda^2}{a}-\frac{1}{2}\right)}\; x'^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{2a}$$

$$.B(x,z)\; B^{-1}(x',z')\; dx'\; dx\; d\lambda_{\frac{a}{2}}(z'),$$

where

$B(x,z) = e^{axz-\frac{a}{2}x^2-\frac{a z^2}{4}}$ and $B^{-1}(x',z') = e^{ax'\overline{z'}-\frac{a}{2}x'^2-\frac{a\overline{z'}^2}{4}}$.

We deduce that the resolvent kernel of H_z^a is $G(z,z',\lambda) =$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}}\left(\frac{a}{2\pi}\; e^{-\frac{a}{2}|z'|^2}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} H(x-x')\frac{x^{\left(-\frac{\lambda^2}{a}-\frac{1}{2}\right)}\; x'^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{2a}$$

$$.B(x,z)\; B^{-1}(x',z')\; dx'\; dx.$$

Using the change of variable $x' = xv$ we get $G(z,z',\lambda) =$

$$\left(\frac{a}{\pi}\right)^{\frac{1}{2}}\left(\frac{a}{2\pi}\; e^{-\frac{a}{2}|z'|^2}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} H(x(1-v))\frac{v^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{2a}$$

$$.B(x,z)B^{-1}(xv,z')\frac{|x|}{x}dvdx$$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}}\left(\frac{a}{2\pi}\; e^{-\frac{a}{2}|z'|^2}\right)\int_{\mathbb{R}}\int_{\mathbb{R}} sgn(v-1)\frac{v^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{2a}$$

$$.B(x,z)\; B^{-1}(xv,z')\; dv\; dx$$

$$= \left(\frac{a}{\pi}\right)^{\frac{1}{2}}\left(\frac{a}{2\pi} e^{-\frac{a}{2}|z'|^2}\right)\int_{\mathbb{R}} sgn(v-1)\frac{v^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{2a}$$

$$.\int_{\mathbb{R}} B(x,z)B^{-1}(xv,z')\; dx\; dv.$$

Let us denote

$$F(z,z',v) = \int_{\mathbb{R}} B(x,z)\; B^{-1}(xv,z')\; dx.$$

Then,

$$F(z,z',v) = \int_{\mathbb{R}} e^{axz-\frac{a}{2}x^2-\frac{a z^2}{4}}\; e^{axv\overline{z'}-\frac{a}{2}x^2v^2-\frac{a\overline{z'}^2}{4}}\, dx$$

$$= \int_{\mathbb{R}} e^{ax(z+v\overline{z'})-\frac{a}{2}x^2} e^{-\frac{a}{2}x^2v^2} e^{-\frac{az^2}{4}-\frac{a\overline{z'}^2}{4}}\, dx$$

$$= e^{-\frac{a}{4}\overline{z'}^2} e^{\frac{a}{4}v^2\overline{z'}^2} e^{\frac{a}{2}vz\overline{z'}} \int_{\mathbb{R}} e^{ax(z+v\,\overline{z'})-\frac{a}{2}x^2} e^{-\frac{a}{2}x^2v^2} e^{-\frac{a}{4}(z+v\overline{z'})^2} dx$$

$$= e^{\frac{a}{4}(v^2-1)\overline{z'}^2}\; e^{\frac{a}{2}vz\overline{z'}}\; \left(\frac{\pi}{a}\right)^{\frac{1}{4}} \left[B_{\frac{a}{2}}\left(e^{-\frac{a}{2}x^2v^2}\right)\right](z+v\overline{z'})$$

Using lemma 2.3, we obtain

$$F(z,z',v) = e^{\frac{a}{4}(v^2-1)\overline{z'}^2} e^{\frac{a}{2}vz\overline{z'}} \left(\sqrt{\frac{\pi}{\frac{a}{2}\,(1+v^2)}}\, e^{\frac{a}{4}\left(\frac{1-v^2}{1+v^2}\right)(z+v\overline{z'})^2}\right)$$

$$= \sqrt{\frac{2\pi}{a}}\frac{1}{\sqrt{1+v^2}}\; e^{\frac{a}{4}(v^2-1)\overline{z'}^2}\; e^{\frac{a}{2}vz\overline{z'}}\; e^{\frac{a}{4}\left(\frac{1-v^2}{1+v^2}\right)(z+v\overline{z'})^2}$$

$$= \sqrt{\frac{2\pi}{a}}\; \frac{1}{\sqrt{1+v^2}}\; e^{\frac{a}{4}\left(\frac{1-v^2}{1+v^2}\right)z^2}\; e^{\frac{a}{4}\left(\frac{v^2-1}{1+v^2}\right)\overline{z'}^2}\; e^{a\left(\frac{v}{1+v^2}\right)z\overline{z'}}$$

Then,

$$G(z,z',\lambda) = \frac{e^{-\frac{a}{2}|z'|^2}}{2\sqrt{2\pi}}\int_{\mathbb{R}} sgn(v-1)\frac{v^{\left(\frac{\lambda^2}{a}-\frac{1}{2}\right)}}{\sqrt{1+v^2}}$$

$$.e^{\frac{a}{4}\left(\frac{1-v^2}{1+v^2}\right)z^2}\; e^{\frac{a}{4}\left(\frac{v^2-1}{1+v^2}\right)\overline{z'}^2}\; e^{a\left(\frac{v}{1+v^2}\right)z\,\overline{z'}}dv. \;\square$$

3 EIGENVECTORS OF DIFFERENTIAL OPERATORS

Let L be a differential operator. We say that $\lambda \in \mathbb{R}$ is an eigenvalue of L if there exists a smooth function $\phi\colon \mathbb{R} \longmapsto \mathbb{R}$ such that $L(\phi) = \lambda\phi$. The function ϕ is called an eigenvector of L. Our aim in this section is to determine the eigenvectors of the real harmonic oscillator defined in (1.9). For this purpose, we compute the eigenvectors of the complex Dirac operator and complex Euler operator defined respectively in (1.7) and (1.8).

Lemma 3.1 The eigenvectors of the complex Dirac operator $\frac{\partial}{\partial z}$ are of the form $\phi(z) = e^{\lambda z}$.

Theorem 3.1 The eigenvectors of the complex Euler operator $E_z^a = -2az\frac{\partial}{\partial z} - a$ are of the form $\phi(z) = z^{\left(\frac{\lambda+a}{-2a}\right)}$.

Proof. Let us suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of E_z^a and ϕ a complex function . Then we have

$-2az\,\phi'(z) - a\phi(z) = \lambda\phi(z).\quad forall\; z \in \mathbb{C}$

which implies that

$-2ae^{-aw}\phi'(e^{-aw}) - a\phi(e^{-aw}) = \lambda\phi(e^{-aw})\; forall\; w \in \mathbb{C}$(3.1)

Let ψ be the complex function defined by

$\psi(w) = \phi(e^{-aw}),\qquad$ forall $w \in \mathbb{C}$.

Then (3.1) becomes

$2\,\psi'(w) - a\psi(w) = \lambda\psi(w),$ forall $w \in \mathbb{C}$.

Using lemma 3.1, we obtain that

$\phi(e^{-aw}) = \psi(w) = e^{\left(\frac{\lambda+a}{2}\right)w} = (e^{-aw})^{\left(\frac{\lambda+a}{-2a}\right)},\qquad$ forall $w \in \mathbb{C}$.

Thus we get

$\phi(z) = z^{\left(-\frac{\lambda+a}{2a}\right)},\qquad$ forall $z \in \mathbb{C}$. □

Theorem 3.2 The eigenvectors of the real Euler operator $e_x^a = ax\frac{\partial}{\partial x} + \frac{a}{2}$ are of the form $\phi(x) = x^{\left(\frac{\lambda}{a}-\frac{1}{2}\right)}$.

Proof. The proof follows mutatis mutandis the proof of theorem 3.1. □

The following proposition is a consequence of lemma 2.2.

Proposition 3.1 Let E_z^a and h_x^a be respectively the complex Euler operator and the real harmonic oscillator given in (1.8) and (1.9). Then, we obtain

$$\left[B_{\frac{a}{2}}(h_x^a f)\right](z) = E_z^a\left[B_{\frac{a}{2}} f\right](z).$$

Theorem 3.3 The eigenvectors of the real harmonic oscillator $h_x^a = \frac{\partial^2}{\partial x^2} - a^2x^2$ are of the form

$$\phi(x) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int_{\mathbb{C}} z^{\left(\frac{\lambda+a}{-2a}\right)}\, e^{ax\bar{z}-\frac{a}{2}x^2-\frac{a}{4}\bar{z}^2}\; d\lambda_a(z).$$

Proof. Let us suppose that $\lambda \in \mathbb{R}$ is an eigenvalue of h_x^a and ϕ a real smooth function. Then

$$\left(\frac{\partial^2}{\partial x^2} - a^2x^2\right)\phi(x) = \lambda\phi(x).$$

Applying the Bargmann transform and using proposition 3.1, we obtain

$$\left(-2az\frac{\partial}{\partial z} - a\right)[B_{\frac{a}{2}}\phi](z) = \lambda\,[B_{\frac{a}{2}}\phi](z).$$

Now theorem 3.1 assures that

$$[B_{\frac{a}{2}}\phi](z) = z^{\left(\frac{\lambda+a}{-2a}\right)}.$$

Then

$$\phi(x) = B_{\frac{a}{2}}^{-1}\left(z^{\frac{\lambda+a}{-2a}}\right)(x) = \left(\frac{2a}{\pi}\right)^{\frac{1}{4}}\int_{\mathbb{C}} z^{\left(\frac{\lambda+a}{-2a}\right)} e^{ax\bar{z}-\frac{a}{2}x^2-\frac{a}{4}\bar{z}^2} d\lambda_a(z).\square$$

4 CONCLUSION

In this paper, we introduce a new method, based on the well-known Bargmann transform, to compute the Green's function and the eigenvectors of harmonic oscillators. This method might be useful for solving a large class of differential equations especially those concerning the operators that can be transported with this integral transform.

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