Secure Domination Polynomials Of Paths

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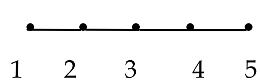
Abstract: In this paper, we study the secure dominating sets of paths and secure domination polynomials of paths in graph theory. We obtain recursive formula for coefficients of secure domination polynomials of paths. Using this recursive formulas, we construct the polynomial which we call secure domination polynomial of path and obtain some properties of this polynomial.

Index Terms: Domination, secure dominating set, secure domination polynomial, secure domination number.

1 INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. The order $|V|$ and the size $|E|$ of G are denoted by n and m respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [2]. For any vertex $v \in V$, the open neighborhood of v is the set $N(v) = \{u \in V : uv \in E\}$ and the closed neighborhood of v is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of S is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set of G, if $N[S] = V$, or equivalently, every vertex in $V - S$ is adjacent to atleast one vertex in S. A dominating set S of G is a secure dominating set if for each $u \in V - S$ there exists $v \in N(u) \cap S$ such that $(S - \{v\}) \cup \{u\}$ is a dominating set. In this case we say that u is S - defended by v or v S - defends u. The secure domination number $\gamma_s(G)$ is the minimum cardinality of a secure dominating set. The concept secure dominating set is introduced by Cockayne et al [3]. A simple path is a path in which all its internal vertices have degree two, and the end vertices have degree one and is denoted by P_n.

Example



P_5 .

Fig 1

Here $V = \{1, 2, 3, 4, 5\}$; $S = \{1, 3, 5\}$ is a dominating set. $V - S = \{2, 4\}$. Then $\{2, 3, 5\}, \{1, 4, 5\}$ are also dominating sets. Therefore $S = \{1, 3, 5\}$ is a secure dominating set.

Cockayane et al [3] obtained a characterization of minimal secure dominating sets.

Definition 1.1[3] Let X be a dominating set of G. Let $S = \{v \in X : X - \{v\} \text{ is a dominating set of } G\}$. For $u \in V - X$, let $A(u, X) = \{v \in X : v\ X - defends\ u\}$.

Theorem 1.2[3] A secure dominating set X is minimal if and only if for each $s \in S$ with $N(s) \cap S \neq \emptyset$, there exists $u_s \in V - X$ such that for each $v \in A(u_s, X) - \{s\}$, one of the following holds:

1. There exists $w \in V - X$ such that $N(w) \cap X = \{v, s\}$ and $u_s \notin N(w)$.

2. $N(s) \cap X = \{v\}$ and $u_s \in N(v) - N(s)$.

Definition 1.3 Let $\mathcal{D}_s(G, i)$ denote the family of all secure dominating set of G with cardinality i. Let $d_s(G, i) = |\mathcal{D}_s(G, i)|$. The secure domination polynomial of $\mathcal{D}_s(G, i) = \sum_{i=\gamma_s(G)}^{|V(G)|} d_s(G, i) x^i$, where $\gamma_s(G)$ is the secure domination number of G.

In the next section, we construct the secure domination polynomials of paths by recursive method. As usual we use $\lfloor x \rfloor$ for the largest integer less than or equal to x and $\lceil x \rceil$ for the smallest integer greater than or equal to x. Also, we denote the set $\{1, 2, \ldots, n\}$ by $[n]$, throughout this paper.

2 MAIN RESULTS

Let $\mathcal{D}_s(P_n, i)$ be the family of secure dominating sets of P_n with cardinality i. Let $d_s(P_n, i) = |\mathcal{D}_s(P_n, i)|$.

Lemma 2.1[3] Let P_n be the path with n vertices. Then $\gamma_s(P_n) = \left\lceil \frac{3n}{7} \right\rceil$ for all n.

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SECURE DOMINATION POLYNOMIAL OF P_n

Let $D_s(P_n, x) = \sum_{i=\lceil \frac{3n}{7} \rceil}^{n} d_s(P_n, i)\, x^i$ be the secure domination polynomial of a path P_n .

Theorem 2.2 Let $\mathcal{D}_s(P_n, i)$ be the family of secure dominating sets with cardinality i and let $d_s(P_n, i) = |\mathcal{D}_s(P_n, i)|$. For every $n \geq 10$,

i) If $n = 7k$ and $i = 3k$ for some $k \in \mathbb{N}$, then

$$d_s(P_n, i) = d_s(P_{n-1}, i-3) + d_s(P_{n-2}, i-3) + d_s(P_{n-3}, i-3) + d_s(P_{n-4}, i-3) + d_s(P_{n-5}, i-3) + d_s(P_{n-6}, i-3) + d_s(P_{n-7}, i-3)$$

.

ii) If $n = 7k+1$ and $i = 3k+1$ for some $k \in \mathbb{N}$, then $d_s(P_n, i) = 2d_s(P_{n-2}, i-1) + d_s(P_{n-9}, i-4)$.

iii) If $n = 7k+2$ and $i = 3k+1$ for some $k \in \mathbb{N}$, then $d_s(P_n, i) = d_s(P_{n-7}, i-3) + d_s(P_{n-2}, i-1)$.

iv) If $n = 7k+3$ and $i = 3k+2$ for some $k \in \mathbb{N}$, then $d_s(P_n, i) = 2d_s(P_{n-2}, i-1) + d_s(P_{n-3}, i-2) - d_s(P_{n-9}, i-4)$

v) If $n = 7k+4$ and $i = 3k+2$ for some $k \in \mathbb{N}$, then $d_s(P_n, i) = d_s(P_{n-2}, i-1) + d_s(P_{n-4}, i-2) + d_s(P_{n-7}, i-3)$

vi) If $n = 7k-1$ and $i = 3k$ for some $k \in \mathbb{N}$, then $d_s(P_n, i) = 2d_s(P_{n-2}, i-1) + d_s(P_{n-7}, i-3) - d_s(P_{n-9}, i-4)$

vii) If $n = 7k-2$ and $i = 3k$ for some $k \in \mathbb{N}$, then $d_s(P_n, i) = 2d_s(P_{n-2}, i-1) + d_s(P_{n-3}, i-2) - d_s(P_{n-9}, i-4)$

viii) Otherwise $d_s(P_n, i) = d_s(P_{n-1}, i-1) + 2d_s(P_{n-2}, i-1) - d_s(P_{n-3}, i-2)$

with initial values $D_s(P_1, x) = x$, $D_s(P_2, x) = x^2 + 2x$,

$D_s(P_3, x) = x^3 + 3x^2$, $D_s(P_4, x) = x^4 + 4x^3 + 4x^2$,

$D_s(P_5, x) = x^5 + 5x^4 + 8x^3$,

$D_s(P_6, x) = x^6 + 6x^5 + 13x^4 + 8x^3$,

$D_s(P_7, x) = x^7 + 7x^6 + 19x^5 + 20x^4 + x^3$,

$D_s(P_8, x) = x^8 + 8x^7 + 26x^6 + 38x^5 + 16x^4$,

$D_s(P_9, x) = x^9 + 9x^8 + 34x^7 + 63x^6 + 48x^5 + 3x^4$.

Theorem 2.3 The following properties hold for the coefficients of $D_s(P_n, x)$

i) $d_s(P_{7n}, 3n) = 1$ for every $n \in \mathbb{N}$.

ii) $d_s(P_n, n) = 1$ for every $n \in \mathbb{N}$.

iii) $d_s(P_n, n-1) = n$ for every $n \geq 2$.

iv) $d_s(P_n, n-2) = \binom{n}{2} - 2$ for every $n \geq 4$.

v) $d_s(P_n, n-3) = \binom{n-2}{n-5} + 2\left(\binom{n-3}{n-5} - 1\right)$ for every $n \geq 6$.

vi) $d_s(P_n, n-4) = \frac{(n-5)(n^3 - n^2 - 42n + 96)}{24} - (3n - 14)$ for every $n \geq 8$.

vii) $d_s(P_{7n+2}, 3n+1) = 2 + n$ for every $n \in \mathbb{N}$.

viii) $d_s(P_{7n+4}, 3n+2) = \frac{n^3 + 7n + 8}{2}$ for every $n \in \mathbb{N}$.

Table-1

$d_s(P_n, i)$, the number of secure dominating set of P_n with cardinality i

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14
n														
1	1													
2	2	1												
3	0	3	1											
4	0	4	4	1										
5	0	0	8	5	1									
6	0	0	8	13	6	1								
7	0	0	1	20	19	7	1							
8	0	0	0	16	38	26	8	1						
9	0	0	0	3	48	63	34	9	1					
10	0	0	0	0	34	104	96	43	10	1				
11	0	0	0	0	8	114	192	138	53	11	1			
12	0	0	0	0	0	74	274	321	190	64	12	1		
13	0	0	0	0	0	20	268	554	501	253	76	13	1	
14	0	0	0	0	0	1	160	702	1004	743	328	89	14	1

Proof:

i) Since $\boldsymbol{D}_s(P_{7n}, 3n) = \{2, 4, \dots, n-3, n-1\}$, we have $d_s(P_{7n}, 3n) = 1$ for every $n \in \mathbb{N}$.

ii) Since $\boldsymbol{D}_s(P_n, n) = \{[n]\}$, we have $d_s(P_n, n) = 1$ for every $n \in \mathbb{N}$.

iii) Since $\boldsymbol{D}_s(P_n, n-1) = \{[n] - \{x\} / x \in [n]\}$, we have $d_s(P_n, n-1) = n$ for every $n \geq 2$.

iv) Proof by induction on n.

Let $n = 4$.

$\text{L.H.S} = d_s(P_4, 2) = 4$ (from table)

$\text{R.H.S} = \binom{4}{2} - 2 = 6 - 2 = 4$

Therefore the result is true for $n = 4$.

Now suppose that the result is true for all numbers less than 'n' and we prove it for n.

By theorem 2.2,

$$d_s(P_n, n-2) = d_s(P_{n-1}, n-3) + 2d_s(P_{n-2}, n-3) - d_s(P_{n-3}, n-4)$$

$$= \binom{n-1}{2} - 2 + 2(n-2) - (n-3)$$

$$= \frac{(n-1)!}{2!(n-3)!} - 2 + 2n - 4 - n + 3$$

$$= \frac{(n-1)(n-2(n-3)!}{2(n-3)!} + n - 1 - 2$$

$$= \frac{2n - 2 + n^2 - 2n - n + 2}{2} - 2$$

$$= \frac{n^2 - n}{2} - 2$$

$$= \frac{n(n-1)}{2} - 2$$

$$= \binom{n}{2} - 2$$

v) By induction on n.

The result is true for $n = 6$.

$\text{L.H.S} = d_s(P_6, 3) = 8$ (from table)

$$\text{R.H.S} = \binom{6-2}{6-5} + 2\left(\binom{6-3}{6-5} - 1\right)$$

$$= \binom{4}{1} + 2\left(\binom{3}{1} - 1\right)$$

$$= 4 + 2(3-1)$$

$$= 8$$

Therefore the result is true for $n = 6$. Now suppose the result is true for all natural numbers less than n.

By theorem 2.2,

$$d_s(P_n, n-3) = d_s(P_{n-1}, n-4) + 2d_s(P_{n-2}, n-4) - d_s(P_{n-3}, n-5)$$

$$= \binom{n-3}{n-6} + 2\left(\binom{n-4}{n-6} - 1\right) + 2\left(\binom{n-2}{2} - 2\right) - \left(\binom{n-3}{2} - 2\right)$$

$$= \frac{(n-3)!}{(n-6)!3!} + 2\left(\frac{(n-4)!}{(n-6)!2!} - 1\right) + 2\left(\frac{(n-2)!}{2!(n-4)!} - 2\right) - \left(\frac{(n-3)!}{2!(n-5)!} - 2\right)$$

$$= \frac{(n-3)(n-4)(n-5)(n-6)!}{(n-6)!3!} + \frac{2(n-4)(n-5)(n-6)!}{(n-6)!2!} - 2 + \frac{2(n-2)(n-3)(n-4)!}{2(n-4)!} - 4 - \frac{(n-3)(n-4)(n-5)!}{2(n-5)!} + 2$$

$$= \frac{(n-3)(n-4)(n-5)(n-6)!}{(n-6)!3!} + \frac{2(n-4)(n-5)(n-6)!}{(n-6)!2!} - 2 + \frac{2(n-2)(n-3)(n-4)!}{2(n-4)!} - 4 - \frac{(n-3)(n-4)(n-5)!}{2(n-5)!} + 2$$

$$= \frac{(n-3)(n-4)(n-5)}{6} + (n-4)(n-5) + (n-2)(n-3) - 4 - \frac{3(n-3)(n-4)}{6}$$

$$= \frac{(n-3)(n-4)(n-8)}{6} - 4 + n^2 - 9n + 20 + n^2 - 5n + 6$$

$$= \frac{(n^2 - 7n + 12)(n-8)}{6} + 2n^2 - 14n + 22$$

$$= \frac{n^3 - 3n^2 - 16n + 36}{6}$$

$$= \binom{n-2}{n-5} + 2\left(\binom{n-3}{n-5} - 1\right)$$

vi) By induction on n.

Let $n = 8$.

$\text{L.H.S} = d_s(P_8, 4) = 16$ (from table)

$$R.H.S = \frac{(8-5)(8^3-8^2-42\times 8+96)}{24}-(3\times 8-14)$$
$$=16$$

Therefore the result is true for $n = 8$.
Now suppose that the result is true for all natural numbers less than or equal to n.
By theorem 2.2,

$$d_s(P_n, n-4) = d_s(P_{n-1}, n-5) + 2d_s(P_{n-2}, n-5) - d_s(P_{n-3}, n-6)$$

$$= \frac{(n-6)((n-1)^3-(n-1)^2-42(n-1)+96)}{24} - (3(n-1)-14)$$
$$+2\left(\binom{n-4}{n-7}+2\left(\binom{n-5}{n-7}-1\right)\right)-\left(\binom{n-5}{n-8}+2\left(\binom{n-6}{n-8}-1\right)\right)$$

$$= \frac{(n-6)((n^3-3n^2+3n-1)-(n^2-2n+1)-42n+138)}{24} - 3n+17$$
$$+2\left(\frac{(n-4)!}{(n-7)!3!}+2\left(\frac{(n-5)!}{(n-7)!2!}-1\right)\right)-\left(\frac{(n-5)!}{(n-8)!3!}+2\left(\frac{(n-6)!}{(n-8)!2!}-1\right)\right)$$

$$= \frac{n^4-10n^3-13n^2+358n-816}{24} - 3n+17+\frac{(n-4)(n-5)(n-6)}{3}$$
$$+2(n-5)(n-6)-4-\frac{(n-5)(n-6)(n-7)}{6}-(n-6)(n-7)+2$$

$$= \frac{n^4-6n^3-37n^2+306n-480}{24} - (3n-14)$$
$$= \frac{(n-5)(n^3-n^2-42n+96)}{24} - (3n-14)$$

vii) By induction on n.
The result is true for $n = 1$.
$L.H.S = d_s(P_9, 4) = 3$ (from table)
$R.H.S = 2+1 = 3$
Therefore the result is true for $n = 1$.

Now suppose the result is true for all natural numbers less than n.
By theorem 2.2,

$$d_s(P_{7n+2}, 3n+1) = d_s(P_{7n+2-7}, 3n+1-3) + d_s(P_{7n+2-2}, 3n+1-1)$$
$$= 2+(n-1)+1$$
$$= 2+n$$

viii) By induction on n.
The result is true for $n-1$.
$L.H.S = d_s(P_{11}, 5) = 8$ (from table)
$$R.H.S = \frac{1^2+7\times 1+8}{2} = 8$$
Therefore the result is true for $n = 1$.

Now suppose the result is true for all natural numbers less than n.
By theorem 2.2,

$$d_s(P_{7n+4}, 3n+2) = d_s(P_{7n+4-2}, 3n+2-1) + d_s(P_{7n+4-4}, 3n+2-2)$$
$$+ d_s(P_{7n+4-7}, 3n+2-3)$$

$$= 2+n+1+\frac{(n-1)^2+7(n-1)+8}{2}$$
$$= 3+n+\frac{n^2-2n+1+7n-7+8}{2}$$
$$= \frac{6+2n+n^2+5n+2}{2}$$
$$= \frac{n^2+7n+8}{2}$$

Theorem 2.4

$$1 = d_s(P_{3n}, 3n) < d_s(P_{3n+1}, 3n) < d_s(P_{3n+2}, 3n) < \ldots\ldots$$
$$< d_s(P_{6n-2}, 3n) < d_s(P_{6n-1}, 3n) > d_s(P_{6n}, 3n) > \ldots\ldots$$
$$> d_s(P_{7n-1}, 3n) > d_s(P_{7n}, 3n) = 1$$

for every $n \ge 2$.
Proof: We will prove that for every n, $d_s(P_i, 3n) < d_s(P_{i+1}, 3n)$ for $3n \le i \le 6n-2$ and $d_s(P_i, 3n) > d_s(P_{i+1}, 3n)$ for $6n-1 \le i \le 7n-1$.
We prove the first inequality by induction on n. The result holds for $n = 1$.
Suppose that result is true for all natural numbers less than n.
Now we will prove it for n, that is, $d_s(P_i, 3n) < d_s(P_{i+1}, 3n)$ for $3n \le i \le 6n-2$.
By theorem 2.2 and induction hypothesis,

$$d_s(P_i, 3n) = d_s(P_{i-1}, 3n-1) + 2d_s(P_{i-1}, 3n-1) - d_s(P_{i-3}, 3n-2)$$
$$< d_s(P_i, 3n-1) + 2d_s(P_{i-1}, 3n-1) - d_s(P_{i-2}, 3n-2)$$
$$= d_s(P_{i+1}, 3n)$$

Similarly, we have the other inequality.

3 CONCLUSION

To conclude the paper it discusses and analyses the secure dominating sets of path and secure domination polynomials of path according to graph theory. Topics in the paper shows about definition, lemmas and theorems related to secure dominating sets of path and secure domination polynomials of path and found the formulas for obtaining the coefficients.

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