Self-Adjoint Operator With Triangular Factorization In Hilbert Space

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Abstract: In this paper we examine and apply the issue of triangular factorization of positive self-adjoint operators in Hilbert space; we demonstrate that expansive classes of operators can be factorized.

Keywords : Triangular operators, operators with difference kernels, operator identity, homogeneous kernels.

1. INTRODUCTION

In the Hilbert space $L_m^2\,(a, a+\epsilon_2)$ we define the orthogonal projectors
$P_{(x+\epsilon)}f = f(x),\ a \le x < (x+\epsilon)$ and $P_{(x-\epsilon_1)}f = 0$, $(x-\epsilon_1) < x \le a+\epsilon_2$, where $f(x) \in L_m^2\,(a, a+\epsilon_2)$.
Definition 1.1. A bounded operator S_- on $L_m^2\,(a, a+\epsilon_2)$ is called lower triangular if for every $(x+\epsilon)$ the relations

$$S_- Q_{(x+\epsilon)} = Q_{\ (x+\epsilon)} S_- Q_{\ (x+\epsilon)} \tag{1.1}$$

are true, where $Q_{(x+\epsilon)} = I - P_{(x+\epsilon)}$.
Definition 1.2. A bounded operator S_+ on $L_m^2\,(a, a+\epsilon_2)$ is called upper triangular if for every $(x-\epsilon_1)$ the relations

$$S_+ P_{(x-\epsilon_1)} = P_{(x-\epsilon_1)} S_+\, P_{(x-\epsilon_1)} \tag{1.2}$$

are true.
Definition 1.3. A bounded, positive and invertible operator S on $L_m^2\,(a, a+\epsilon_2)$ is said to admit the right triangular factorization if it can be represented in the form

$$S = {S_+}^2 \tag{1.3}$$

where S_+ and S_+^{-1} are upper triangular, bounded self-adjoint operators.
Definition 1.4. A bounded, positive and invertible operator S on $L_m^2\,(a, a+\epsilon_2)$ is said to admit the left triangular factorization if it can be represented in the form

$$S = {S_-}^2 \tag{1.4}$$

where S_- and S_-^{-1} are lower triangular, bounded self-adjoint operators.
1. Gohberg and M.G. Krein [4] studied the problem of factorization under the assumption

$${S_-}^2 - I \in \gamma_\infty \tag{1.5}$$

where γ_∞ is the set of compact operators. The operators S_- and S_+ were assumed to have the form $\sqrt{S} = I + X_+$, $\sqrt{S} = I + X_-;\ X_+, X_- \in \gamma_\infty$. The factorization method plays an important role in a number of analysis problems. The factorizing operator $V = S_-^{-1}$ is constructed in an explicit form, also he consider the class of positive operators S which satisfy the operator identity (See [1,2,3,5,6,8,9,11]).
2. TRIANGULAR FACTORIZATION WITH SELF-ADJOINT OPERATOR
Let S_+^2 be a linear, bounded, self-adjoint and invertible operator S^* on $L_m^2(a, a+\epsilon_2)$.We introduce the notation

$$[S_+^2]_{(x+\epsilon)} = P_{(x+\epsilon)} S_+^2 P_{(x+\epsilon)},\ \ (f,g)_{(x+\epsilon)} = \int_a^{(x+\epsilon)} g^*(x) f(x)dx \tag{2.1}$$

where $f(x), g(x) \in L_m^2(a, a+\epsilon_2)$.
We show the following theorem [11].
Theorem 2.1 Let the bounded and invertible operator S_+^2 on $L_m^2(a, a+\epsilon_2)$.be positive. For the self-adjoint operator S_+^2 to admit the left triangular factorization it is necessary and sufficient that the following assertions are true.
1. There exists an $m \times m$ matrix function $F_0(x)$ such that

$$Tr \int_a^{a+\epsilon_2} |F_0^*(x)|^2 dx < \infty, \tag{2.2}$$

that the $m \times m$ matrix function

$$M(x+\epsilon) = \left(F_0^*(x), [S^*]_{(x+\epsilon)}^{-1}\, F_0^*(x)\right)_{(x+\epsilon)} \tag{2.3}$$

is absolutely continuous, and almost everywhere

$$\det \acute{M}(x+\epsilon) \neq 0. \tag{2.4}$$

2. THE VECTOR FUNCTIONS

$$\int_a^x v^*(x,t) f(t) dt \tag{2.5}$$

are absolutely continuous. Here $f(x) \in L_m^2(a, a+\epsilon_2)$ and

$$v((x+\epsilon), t) = [S^*]_{(x+\epsilon)}^{-1}\, P_{(x+\epsilon)} F_0^*(x), \tag{2.6}$$

(In (2.3) the self -adjoint operator $[S^*]_{(x+\epsilon)}^{-1}$ transforms the matrix column of the original into the corresponding column of the image.)

3. THE OPERATOR

$$V^* f = [R^*(x)]^{-1} \frac{d}{dx} \int_0^x v^*(x,t) f(t) dt \tag{2.7}$$

is bounded, invertible and lower triangular with its inverse $[V^*]^{-1}$. Here $R^*(x)$ is an $m \times m$ matrix function such that

$$[R^*(x)]^2 = \acute{M}(x). \tag{2.8}$$

Proof. Necessity. We suppose that the self-adjoint operator S^* admits the left triangular factorization (1.4). Let $F_0^*(x) \in L_m^2(a, a+\epsilon_2)$ be a fixed $m \times m$ matrix function satisfying relation (2.2). We introduce the $m \times m$ matrix function

$$R^*(x) = V^* F_0^*(x), \tag{2.9}$$

where $V^* = [S^*]^{-2}$. We can choose $F_0^*(x)$ in such a way that almost everywhere then equality

$$\det R^*(x) \neq 0 \tag{2.10}$$

is true.
From relations (1.4), (2.3) and (2.9) we have

$$M(x+\epsilon) = \int_a^{(x+\epsilon)} |R^*(x)|^2 dx. \tag{2.11}$$

Hence the function $M(x+\epsilon)$ is absolutely continuous and

$$\acute{M}(x) = [R^*(x)]^2. \tag{2.12}$$

Now we use the equality

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$$\left(f,\,[S^*]^{-1}_{(x+\epsilon)}\,F_0^*\right)_{(x+\epsilon)} = (V^*f, V^*F_0^*)_{(x+\epsilon)}. \quad (2.13)$$

Relations (2.9) and (2.13) imply that

$$\frac{d}{dx}\int_a^x v^*(x,t)f(t)dt = R^*(x)(V^*f). \quad (2.14)$$

The necessity is proved.

Sufficiency. Let the conditions 1–3 of Theorem 2.1 be fulfilled. It follows from (2.6)–(2.8) that

$$V^*F_0^* = R^*(x). \quad (2.15)$$

We can write $\acute{M}(x) = [V^*F_0^*]^2$.

From relations (2.6), (2.7) and (2.15) we deduce that

$$(V^*f, V^*F_0^* =)_{(x+\epsilon)} = (f, [S^*]^{-1}_{(x+\epsilon)}\,P_{(x+\epsilon)}F_0^*)_{(x+\epsilon)}, \quad i.e.,$$

$$V^*P_{(x+\epsilon)}V^*\,P_{(x+\epsilon)}F_0 = [S^*]^{-1}_{(x+\epsilon)}\,P_{(x+\epsilon)}F_0^*. \quad (2.16)$$

We define $v((x-\epsilon_1),t)$ in the domain $(x-\epsilon_1) \le t \le a+\epsilon_2$ by the equality $v((x-\epsilon_1),t) = 0$. It follows from the triangular structure of the self -adjoint operators V^* and $[V^*]^{-1}$ that

$$P_{(x-\epsilon_1)}[V^*]^{-1}\,P_{(x-\epsilon_1)}V^*\,P_{(x-\epsilon_1)} = P_{(x-\epsilon_1)}. \quad (2.17)$$

Hence in view of (2.6) and (2.16) we have

$$P_{(x-\epsilon_1)}[V^*]^{-2}v((x-\epsilon_1),t) = P_{(x-\epsilon_1)}F_0^*. \quad (2.18)$$

It is easy to see that $P_{(x-\epsilon_1)}S^*v((x-\epsilon_1),t) = P_{(x-\epsilon_1)}F_0^*$. Thus according to relations (2.17) and (2.18), the equality

$$\left([V^*]^{-2}v\big((x-\epsilon_1),t\big), v(\mu,t)\right) = \left(S^*v\big((x-\epsilon_1),t\big), v(\mu,t)\right) \quad (2.19)$$

is true. If there exists such a vector function $f_0(x) \in L_m^2(a,a+\epsilon_2)$ that $(f_0, v((x-\epsilon_1),t)) = 0$,then due to (2.7) the relation

$$V^*f_0 = 0 \quad (2.20)$$

Is valid. The self -adjoint operator V^* is invertible. Hence from (2.20) we deduce that $f_0 = 0$. This means that $v((x-\epsilon_1),t)$ is a complete system in $L_m^2(a,a+\epsilon_2)$ Using this fact and relation (2.19) we obtain the desired equality

$$S^* = [V^*]^{-2}. \quad (2.21)$$

The theorem is proved. □

Corollary 2.2. If the conditions of Theorem 2.1 and Theorem 2.3 are fulfilled, then the corresponding self -adjoint operator $[S^*]^{-1}$ can be represented in the form

$$[S^*]^{-1} = [V^*]^2 = U^*U. \quad (2.22)$$

We introduce the notation

$$C_{(x-\epsilon_1)} = Q_{(x-\epsilon_1)}S^*Q_{(x-\epsilon_1)},$$

$$[f,g]_{(x-\epsilon_1)} = \int_{(x-\epsilon_1)}^{a+\epsilon_2} g^*(x)f(x)dx. \quad (2.23)$$

In the same way as Theorem (2.1) we deduce the following result (See [11]).

Theorem 2.3. Let the bounded and invertible self-adjoint operator S^* on $L_m^2(a,a+\epsilon_2)$ be positive. For the self-adjoint operator S^* to admit the right triangular factorization it is necessary and sufficient that the following assertions are true.

1. There exists an $m\times m$ matrix function $F_0^*(x)$ such that

$$Tr\int_a^{a+\epsilon_2} |F_0^*\,(x)|^2dx < \infty, \quad (2.24)$$

that the $m\times m$ matrix function

$$N(x-\epsilon_1) = \left[F_0^*(x), C^{-1}_{(x-\epsilon_1)}\,F_0^*(x)\right]_{(x-\epsilon_1)} \quad (2.25)$$

is absolutely continuous, and almost everywhere

$$det\acute{N}(x-\epsilon_1) \neq 0. \quad (2.26)$$

2. The vector functions

$$\int_x^{a+\epsilon_2} u^*(x,t)f(t)dt \quad (2.27)$$

are absolutely continuous. Here $f(x) \in L^2(a,a+\epsilon_2)$ and

$$u((x-\epsilon_1),t) = C^{-1}_{(x-\epsilon_1)}\,Q_{(x-\epsilon_1)}F_0^*. \quad (2.28)$$

3. The operator

$$Uf = -[Q^*(x)]^{-1}\frac{d}{dx}\int_x^{a+\epsilon_2} u^*(x,t)f(t)dt \quad (2.29)$$

is bounded, upper triangular and invertible together with its inverse U^{-1}.

Here

$$Q^*(x)Q(x) = -\acute{N}(x). \quad (2.30)$$

The connection between the self-adjoint operators V^* and W is given by the following assertion (See [11]).

Proposition 2.4. Let the self-adjoint operator V^* defined by formula (2.7) be bounded. Then the operator W is also bounded and

$$WT^* = V^*. \quad (2.34)$$

Proof. It can be proved by linear algebra methods that (see [9])

$$T^*Q_{(x-\epsilon_1)}[T^*]^{-1}_{(x-\epsilon_1)}\,Q_{(x-\epsilon_1)}T^* = T^* - [S^*]^{-1}_{(x-\epsilon_1)}\,P_{(x-\epsilon_1)}. \quad (2.35)$$

From relations (2.6), (2.32) and (2.35) we have

$$T^*w\big((x-\epsilon_1),t\big) = T^*F_0^* - v\big((x-\epsilon_1),t\big). \quad (2.36)$$

Hence the equality

$$[T^*f, w((x-\epsilon_1),t)]_{(x-\epsilon_1)} = (T^*f, F_0^*) - \left(f, v\big((x-\epsilon_1),t\big)\right)_{(x-\epsilon_1)} \quad (2.37)$$

is true. From formulas (2.7), (2.33) and (2.37) we obtain relation (2.34). The proposition is proved.

Using Proposition 2.4 we deduce [11] the following important assertion.

Proposition 2.5. Let S^* be a bounded, positive, self-adjoint and invertible operator and let the operator V^* defined by formula (2.7) be bounded. If the relations

$$V^*\,F_0^* = R^*(x), \quad (2.38)$$

and

$$V^*f \neq 0, \ \ \|f\| \neq 0 \quad (2.39)$$

are true, then the self-adjoint operator V^* is invertible, the operator $[V^*]^{-1}$ is lower triangular, and

$$T^* = [V^*]^2. \quad (2.40)$$

Proof. The proof is clear from (2.34) and (2.50).

Corollary2.6. Suppose the hypothesis of Propositions 2.2 and 2.4 are satisfied

(i) $WT^*F_0^* = R^*(x)$.

(ii) $W[U^*U]F_0^* = R^*(x)$ and hence $T^* = [U^*]^2$.

(iii) $[WT^*F_0^*]^2 = \acute{M}(x)$.

Proof:

(i) since $WT^* = V^*$,$WT^*F_0^* = V^*F_0^* = R^*(x)$.

(ii) $V^*F_0^* = WT^*F_0^* = W[S^*]^{-1}F_0^* = W[U^*U]F_0^*$, which implied that $T^* = [U^*]^2$.

(iii) Since

$$\acute{M}(x) = [R^*(x)]^2 = [V^*F_0^*]^2 = [WT^*F_0^*]^2 .$$

Example 2.7. Let us consider the operator

$$S^*f = f(x) + \frac{i}{\pi}V^*.P.\int_a^{a+\epsilon_2} f(t)\frac{c(t)c(x)}{x-t}dt, \quad -\infty < a < a+\epsilon_2 < \infty, \quad (2.38)$$

where $0 < m < c(t) < 1$. The operator (2.38) does not satisfy condition (1.5) but admits the left triangular factorization (see [7]).

3. Self-adjoint Operator and factorization problems

We consider (See [11]) the self-adjoint operators A^*, S^*, Π^* and J satisfying the operator identity

$$A^*S^* - S^*A^* = iJ\Pi^{*2}. \qquad (3.1)$$

We suppose that the self-adjoint operators $A^*$and S^* act on the Hilbert space $L_m^2(0, a+\epsilon_2)$, the self-adjoint operator Π^* maps G ($\dim G = n < \infty$) into $L_m^2(0, a+\epsilon_2)$, the operator J acts on G, and $J = J^*$, and $J^2 = I_n$. We note that the operator Π^* has the form $\Pi^* g = [\phi_1(x), \phi_2(x), \dots, \phi_n(x)]g$, where $\phi_k(x)$ are $m\times 1$ vector functions, $g = col[g_1, g_2, \dots, g_n]$, $\phi_k(x) \in L_m^2(0, a+\epsilon_2)$, Relation (3.1) is fulfilled for the self-adjoint operators S^* which play an important role in the spectral theory of the canonical differential systems (see [9,11]).

Theorem 3.1. Let the following conditions be fulfilled.

1. The self-adjoint operator S^* is bounded, positive and invertible.
2. The relations

$$A^*P_{(x-\epsilon_1)} = P_{(x-\epsilon_1)}A^*P_{(x-\epsilon_1)}, \quad 0 \le (x-\epsilon_1) \le a+\epsilon_2 \quad (3.2)$$

are true.

3. The spectrum of the self-adjoint operator A^* is concentrated at the origin and there is a constant $M > 0$ such that

$$\left\|\left(P_{(x-\epsilon_1)+\Delta(x-\epsilon_1)} - P_{(x-\epsilon_1)}\right)A^*\left(P_{(x-\epsilon_1)+\Delta(x-\epsilon_1)} - P_{(x-\epsilon_1)}\right)\right\| \le M|\Delta(x-\epsilon_1)|, \quad 0 \le (x-\epsilon_1) \le a+\epsilon_2. \quad (3.3)$$

Then the $n\times n$ matrix function

$$W\big((x-\epsilon_1), z\big) = I_n + izJS^{*-1}_{(x-\epsilon_1)}(I - zA^*)^{-1}P_{(x-\epsilon_1)}\Pi^{*2} \quad (3.4)$$

satisfies the matrix integral equation

$$W(x,z) = I_n + izJ\int_0^x [dB(t)]W(t,z)\,, \qquad (3.5)$$

where

$$B(x+\epsilon) = [\Pi^*]^2[S^*]^{-1}_{(x+\epsilon)}P_{(x+\epsilon)}. \qquad (3.6)$$

From relations (1.4) and (3.6) we obtain (See [11]) the necessary conditions for the self-adjoint operator S^* to admit the left triangular factorization.

Proposition 3.2. Let the self-adjoint operator S^* satisfy the relation (3.1) and let the conditions of Theorem 3.1 be fulfilled. If the self-adjoint operator S^* admits the left triangular factorization, then the matrix function $B(x)$ is absolutely continuous and

$$\frac{d}{dx}B(x) = H(x) = [\beta^*(x)]^2, \qquad (3.7)$$

where

$$\beta^*(x) = [h_1(x), h_2(x), \dots, h_n(x)], \;\; h_k(x) = V^*\phi_k(x), V^* = [S^*]_-^{-1}. \qquad (3.8)$$

Using relations (3.5) and (3.7) we obtain that

$$\frac{d}{dx}W(x,z) = izJH(x)W(x,z). \qquad (3.9)$$

Lemma 3.3. Let the conditions of Proposition 3.2 be fulfilled and let the $m\times 1$ vector functions

$$F_j(x,z) = \left(I - A^*z\right)^{-1}\phi_j\,, \quad 1 \le j \le n \quad (3.10)$$

form a complete system in $L_m^2(a, a+\epsilon_2)$. Then we have the equality

$$\mathrm{mes}E = 0, \qquad (3.11)$$

where the set E is defined by the relation

$$x \in E \qquad if \;\; H(x) = 0. \qquad (3.12)$$

Proof. We use the following relation (see [9]):

$$\frac{J - W^*\big((x+\epsilon),\mu\big)JW\big((x+\epsilon),\lambda\big)}{i(\bar\mu - \lambda)} = \Pi^*(I - \bar\mu A^*)^{-1}[S^*]^{-1}_{(x+\epsilon)}(I - \lambda[A^*]^{-1}P_{(x+\epsilon)}\Pi). \qquad (3.13)$$

Formula (3.13) implies that

$$\left([S^*]^{-1}_{(x+\epsilon)}F_j(x,\lambda), F_\ell(x,\mu)\right)_{(x+\epsilon)} = \frac{i\left[Y_\ell^*\left((x+\epsilon),\mu\right)JY_j\big((x+\epsilon),\lambda\big) - Y_\ell^*\;(0,\mu)JY_j(0,\lambda)\right]}{\bar\mu - \lambda}, \qquad (3.14)$$

Where

$$Y_j(x,\lambda) = col\left[W_{1,j}(x,\lambda), W_{2,j}(x,\lambda), \dots W_{n,j}(x,\lambda)\right].$$

Here $W_{i,j}(x,\lambda)$ are entries of $W(x,\lambda)$. In view of (3.19) and (3.14) we have

$$\frac{d}{d(x+\epsilon)}\left([S^*]^{-1}_{(x+\epsilon)}F_j(x,\lambda), F_\ell(x,\mu)\right)_{(x+\epsilon)} = 0, (x+\epsilon) \in E. \qquad (3.15)$$

From (3.12) and (3.15) it follows that

$$\frac{d}{d(x+\epsilon)}(V^*F_j(x,\lambda), V^*F_\ell(x,\mu))_{(x+\epsilon)} = 0, (x+\epsilon) \in E, \quad (3.16)$$

i.e., the relation

$$\left[V^*F_j\right](x,\lambda) = 0 \;, x \in E, \quad 1 \le j \le n, \quad (3.17)$$

is true.

As the self-adjoint operator V^* is invertible and the system of functions $F_j(x,\lambda)$ is complete in $L_m^2(0, a+\epsilon_2)$, the system of the functions $V^*F_j(x,\lambda)$ is also complete in $L_m^2(0, a+\epsilon_2)$The assertion of the lemma follows from this fact and equality (3.17). □

Further we suppose that the $n\times n$ matrix function $B(x)$ is absolutely continuous and that relations (3.7), (3.8) are true [11]. Let us introduce the $m\times m$ matrix functions

$$R^*(x) = h_1(x)\alpha_1 + h_2(x)\alpha_2 + \cdots + h_n(x)\alpha_n, \qquad (3.18)$$

$$F_0^*(x) = \phi_1(x)\alpha_1 + \phi_2(x)\alpha_2 + \cdots + \phi_n(x)\alpha_n, \qquad (3.19)$$

$$v((x+\epsilon), x) = [S^*]^{-1}_{(x+\epsilon)}P_{(x+\epsilon)}F_0^*(x), \qquad (3.20)$$

where α_k are constant $1\times m$ matrices. From Proposition (3.2) we deduce:

Corollary 3.4. Let the conditions of Theorem 3.1and Lemma 3.3 be fulfilled. If $m = 1$, then there exist numbers $\alpha_1, \alpha_2, \dots, \alpha_n$ such that almost everywhere we have the inequality

$$R(x) \ne 0. \qquad (3.21)$$

Now we can verify the formulation of the result obtained by [11].

Theorem 3.5. Let the following conditions be fulfilled.

1. The self-adjoint operator S^* satisfies relation (3.1).
2. The conditions of Theorem 3.1 are valid.
3. The matrix function $B(x)$ is absolutely continuous and formulas (3.7) and (3.8) are true.
4. The vector functions $F_j(x,\lambda)$ $(1 \le j \le n)$ form a complete system in $L_m^2(a, a+\epsilon_2)$.
5. Almost everywhere the inequality

$$\det R^*(x) \ne 0 \qquad (3.22)$$

holds. Then the self-adjoint operator $T^* = [S^*]^{-1}$ admits the right triangular factorization

Proof. We introduce the self-adjoint operator

$$V^*f = [R^*(x)]^{-1}\frac{d}{dx}\int_0^x v^*(x,t)f(t)dt\,. \qquad (3.23)$$

From (3.4), (3.22) and (3.23) we deduce the equality

$$V^*F_j = [h_1(x), \dots, h_n(x)]Y_j(x,z). \qquad (3.24)$$

Relation (3.24) implies that

$$\left(V^* F_j(x,\lambda), V^* F_\ell(x,\mu)\right) = \int_0^{a+\epsilon_2} Y_\ell^*(x,\mu) H(x) Y_j(x,\lambda) dx . \quad (3.25)$$

Using equality (3.24) and relation

$$\frac{d}{dx} Y_j(x,z) = izJH(x)Y_j(x,z) \quad , \quad (3.26)$$

we have

$$\left(V^* F_j(x,\lambda), V^* F_\ell(x,\mu)\right) = \frac{i\left[Y_\ell^*(a+\epsilon_2,\mu)JY_j(a+\epsilon_2,\lambda) - Y_\ell^*(0,\mu)JY_j(0,\lambda)\right]}{\bar{\mu}-\lambda} \quad (3.27)$$

Comparing formulas (3.14) and (3.27) we obtain the equality

$$T^* = [V^*]^2. \quad (3.28)$$

This means that the introduced self-adjoint operator V^* is bounded, $V^* f \neq 0$, and $\|f\| \neq 0$. Taking into account (3.18), (3.19) and (3.24) when $z = 0$ we obtain the relation

$$V^* F_0^* = R^*. \quad (3.29)$$

Thus all conditions of Proposition 2.7 are fulfilled. The assertion of the theorem follows from Proposition.2.7. □

Proposition 3.6. Let the following conditions be fulfilled.

1. Conditions 1–3 of Theorem 3.5 are valid.
2. The $m \times m$ blocks $b_{1,j}(x)$ $(1 \leq j \leq n)$ of the matrix $B(x)$ are absolutely continuous and

$$b_{1,j}(x) = h_1^*(x)h_j(x). \quad (3.30)$$

3. All the entries of the matrices $h_j(x)$ belong to $L^2(a, a+\epsilon_2)$.
4. Almost everywhere the inequality (3.22) holds. Here $R^*(x) = h_1(x)$. Then the self-adjoint operator V^* defined by formula (3.23) and the equality

$$v\left((x-\epsilon_1), x\right) = [S^*]^{-1}_{(x-\epsilon_1)} P_{(x-\epsilon_1)} \varphi_1(x) \quad (3.31)$$

are bounded.

Proof. We introduce the matrix $H(x) = [\beta^*(x)]^2$ where $\beta^*(x) = [h_1(x), h_2(x), \ldots, h_n(x)]$. Relations (3.23)–(3.25) remain true. We use the formula

$$\int_0^{a+\epsilon_2} Y_\ell^*(x,\mu)[dB(x)]Y_j(x,\lambda)dx = \frac{i\left[Y_\ell^*(a+\epsilon_2,\mu)JY_j(a+\epsilon_2,\lambda) - Y_\ell^*(0,\mu)JY_j(0,\lambda)\right]}{\bar{\mu}-\lambda} \quad (3.32)$$

And the inequality $H(x)dx \leq dB(x)$. From formulas (3.14), (3.25) and (3.32) we deduce that

$$[V^*]^2 \leq T. \quad (3.33)$$

The proposition is proved. □

4. Self-adjoint operators with sum-difference Kernels

Let us consider the bounded, positive, self-adjoint and invertible operator S^* with the difference kernel

$$S^* f = \frac{d}{dx}\int_0^a f(t)s(x-t)dt. \quad (4.1)$$

Let us put

$$A^* f = i\int_0^x f(t)dt, \qquad f \in L^2(0,a). \quad (4.2)$$

Equality (3.1) is valid (see [8]).

If

$$J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , \quad (4.3)$$

$$\phi_1(x) = M(x), \quad \phi_2(x) = 1, \quad (4.4)$$

where $M(x) = s(x)$, $0 \leq x \leq a$. In the case under consideration the matrix $B(x+\epsilon)$ has the form

$$B(x+\epsilon) = \begin{bmatrix} \left([S^*]^{-1}_{(x+\epsilon)} M, M\right) & \left([S^*]^{-1}_{(x+\epsilon)} 1, M\right) \\ \left([S^*]^{-1}_{(x+\epsilon)} M, 1\right) & \left([S^*]^{-1}_{(x+\epsilon)} 1, 1\right) \end{bmatrix}. \quad (4.5)$$

The corresponding function $F(x,\lambda)$ has the form

$$F(x,\lambda) = e^{ix\lambda}. \quad (4.6)$$

The self-adjoint operator A^* defined by formula (4.2) satisfies all the conditions of Theorem 3.1 The following fact is useful here. (See [11]).

Theorem 4.1. Let the self-adjoint operator S^* be bounded, positive, invertible and have the form (4.1). If the matrix function $B(x)$ is absolutely continuous and

$$\acute{B}(x) = \beta^*(x)\beta(x), \ \beta(x) = [h_1(x), h_2(x)], \quad (4.7)$$

Then the equality

$$h_1(x)\overline{h_2(x)} + h_2(x)\overline{h_1(x)} = 1 \quad (4.8)$$

is true almost everywhere.

Proof. Let us consider the expression

$$i_{(x+\epsilon)} = \left([S^*]^{-1}_{(x+\epsilon)} P_{(x+\epsilon)} M, 1\right) + \left(1, [S^*]^{-1}_{(x+\epsilon)} P_{(x+\epsilon)} M\right). \quad (4.9)$$

Setting

$$N_1(x,(x+\epsilon)) = [S^*]^{-1}_{(x+\epsilon)} P_{(x+\epsilon)} M, \quad (4.10)$$

we rewrite formula (4.9) in the form $i_{(x+\epsilon)} = \int_0^{(x+\epsilon)}[N_1(x,(x+\epsilon)) + \overline{N_1(x,(x+\epsilon))}]dx$, i.e.,

$$i_{(x+\epsilon)} = \int_0^{(x+\epsilon)} \left[N_1\left(x,(x+\epsilon)\right) + \overline{N_1\left((x+\epsilon)-x,(x+\epsilon)\right)}\right] dx . \quad (4.11)$$

We use the relation (see [8])

$$N_1\left(x,(x+\epsilon)\right) + \overline{N_1\left((x+\epsilon)-x,(x+\epsilon)\right)} = 1. \quad (4.12)$$

In view of (4.11) and (4.12) we obtain the equality

$$i_{(x+\epsilon)} = (x+\epsilon). \quad (4.13)$$

Taking into consideration Equalities (2.1), (3.8), (4.1) and (4.9) we deduce that

$$i_{(x+\epsilon)} = \int_0^{(x+\epsilon)} [h_1(x)\overline{h_2(x)} + h_2(x)\overline{h_1(x)}]dx . \quad (4.14)$$

Relation (4.8) follows from (4.13) and (4.14). The theorem is proved. □

From equality (4.8) we have

$$h_2(x) \neq 0, \quad 0 \leq x \leq a. \quad (4.15)$$

Theorem 4.2. Let the self-adjoint operator S^* be positive, invertible and have the form (4.1). The self-adjoint operator S^* admits the left triangular factorization if and only if the matrix $B(x)$ is absolutely continuous and relation (4.7) is valid.

Example 4.3. Let us consider (See [11]) the self-adjoint operator S_β^* of the form

$$S_\beta^* f = f + \frac{i\beta}{\pi} V^*.P.\int_0^{a+\epsilon_2} \frac{f(t)}{x-t}dt, \quad (4.16)$$

where $-1 < \beta < 1$. This operator with a difference kernel is bounded, invertible and positive (see [7]). The self-adjoint operator S_β^* does not satisfy condition (1.4). Nevertheless S_β^* admits the left triangular factorization $S_\beta^* = W_\alpha W_\alpha^*$, where

$$W_\alpha f = \frac{x^{i\alpha}}{\sqrt{ch(\pi\alpha)}\,\Gamma(i\alpha-1)}\frac{d}{dx} \int_0^x f(t)(x-t)^{-i\alpha}dt. \quad (4.17)$$

Here $\alpha = \frac{1}{\pi}\operatorname{arcth}\beta$, and $\Gamma(z)$ is the gamma function.

We consider the following class of bounded, self-adjoint and positive operators which can be represented in the form $((+,-)-\text{class})$:

$$S^*f = \frac{d^2}{dx^2}\int_0^{a+\epsilon_2}[s_1(x-t)+s_2(x+t)]f(t)dt, \quad (4.18)$$

where $f(t) \in L^2(0,a+\epsilon_2)$. We introduce the self-adjoint operator

$$A^*f = \int_0^x (t-x)f(t)dt. \quad (4.19)$$

Then the operator identity (3.1) is valid. Here the 4×4 matrix J is defined by the relation

$$J = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}, \quad (4.20)$$

and the operator Π has the form

$$\Pi = [\Phi_1, \Phi_2], \quad (4.21)$$

the operators Φ_1 and Φ_2 are defined by the relations

$$\Phi_1 g = -iM(x)g_1 - iM_0(x)g_2, \quad (4.22)$$

$$\Phi_2 g = g_1 + xg_2, \quad (4.23)$$

where

$$M(x) = -[s_1(x)+s_2(x)],\ M_0(x) = \acute{s}_1(x) - \acute{s}_2(x), \quad (4.24)$$

and a constant 2×1 vector g has the form $g = \mathrm{col}[g_1, g_2]$. The main result of this section follows directly from Proposition 3.2, Lemma 3.3 and Theorem 3.5.

We show the following:

Theorem 4.4. Let the self-adjoint operator S^* be positive, invertible and have the form (4.18). The operator S^* admits the left triangular factorization if and only if the matrix $B(x)$ is absolutely continuous and

$$\acute{B}(x) = \beta^*(x)\beta(x),$$

$$\beta(x) = [h_1(x), h_2(x), h_3(x), h_4(x)]. \quad (4.25)$$

Example 4.5. Let us consider the equation

$$S^*f = f(x) + \frac{i\mu}{\pi}V^*.P.\int_0^1 \frac{f(t)}{x-t}dt - \frac{\lambda}{\pi}\int_0^1 \frac{f(t)}{x+t}dt = g(x), \quad (4.26)$$

where $f(x) \in L^2(0,1)$, $\lambda = \bar{\lambda}$, $\mu = \overline{\mu}$, and $|\lambda| + |\mu| < 1$. It is well known ([4]) that the self-adjoint operator S^* is bounded, positive and invertible, i.e., the operator S belongs to the $(+,-)$ class . We introduce the functions

$$v(x,\lambda,\mu) = [S^*]^{-1}1, \alpha(\lambda,\mu) = \int_0^1 v(x,\lambda,\mu)dx = ([S^*]^{-1}1, 1) > 0. \quad (4.27)$$

In view of (4.26) and (4.27) the relations

$$[S^*]^{-1}_{(x-\epsilon_1)}P_{(x-\epsilon_1)}1 = v\left(\frac{x}{(x-\epsilon_1)}, \lambda, \mu\right), \quad \left([S^*]^{-1}_{(x-\epsilon_1)}P_{(x-\epsilon_1)}1, 1\right)_{(x-\epsilon_1)} = (x-\epsilon_1)\alpha(\lambda,\mu) \quad (4.28)$$

are true. We introduce the self-adjoint operator

$$V^*f = \frac{1}{\sqrt{\alpha(\lambda,\mu)}}\frac{d}{dx}\int_0^x f(t)v(\frac{t}{x},\lambda,\mu)dt. \quad (4.29)$$

Using Proposition 3.6. we deduce that the operator V^* is bounded and $[S^*]^{-1} \geq [V^*]^2$.

5. Triangular Factorization, Class $\mathbf{R}_1^*$

Let us consider the integral operators

$$A^*f = i\int_0^x f(t)dt, \quad A^*f = -i\int_x^{a+\epsilon_2} f(t)dt, \quad (5.1)$$

where $f(x) \in L^2(0,a+\epsilon_2)$.

Definition 5.1. We say that the linear bounded operator S^* acting in the Hilbert space $L^2(0,a+\epsilon_2)$. belongs to the class R_1^* (rank 1) if the following conditions are fulfilled:

1) $m(f,f) \leq (Sf,f) \leq M(f,f), \quad 0 < m < M < \infty,$ (5.2)

2) $rank(AS - SA^*) = 1$, i.e.,

$$(AS - SA^*)f = i(f,\phi)\phi, \quad \phi(x) \in L^2(0,a+\epsilon_2). \quad (5.3)$$

We associate with the operator S the operator

$$S_-f = \frac{d}{dx}\int_0^x f(t)\phi(x-t)dt. \quad (5.4)$$

It is easy to see that

$$S_-1 = \phi. \quad (5.5)$$

Note that if $A^*S^* = S^*A^*$ then the $rank = 0$

Lemma 5.2. Let the bounded self-adjoint operator S^* satisfy relation (5.3). If the corresponding operator S_-^* is bounded, then the representation

$$S^* = [S_-^*]^2 \quad (5.6)$$

is true.

Proof. We consider the operator

$$X = [S_-^*]^2. \quad (5.7)$$

Using formula (5.3) and relation $A^*S_-^* = S_-^*A^*$ we deduce the equality

$$A^*X - XA^* = S_-^*(A^*-A^*)S_-^* = A^*S^* - S^*A^*. \quad (5.8)$$

The equation $A^*X - XA^* = F^*$ has no more than one solution X (see [8]). We can deduce that $A^*X = XA^*$ and $F^* = 0$. Hence we deduce from (5.8) that $S^* = X$. The lemma is proved . □

We show the following lemma (See [11]):

Lemma 5.3. If the bounded self-adjoint operator S^* satisfies the relation (5.3), then this operator can be represented in the form (5.6), where the operator S_-^* is defined by formula (5.4).

Proof. To prove that the self-adjoint operator S_-^* is bounded we introduce the operator

$$X_-f = A^*S_-^*f = i\int_0^x f(t)\phi(x-t)dt. \quad (5.9)$$

We note that

$$X_-^*f = S_-^*A^*f = -i\int_x^{a+\epsilon_2} f(t)\overline{\phi(t-x)}dt \quad (5.10)$$

where the operator S_-^* has the form

$$S_-^*f = -\frac{d}{dx}\int_x^{a+\epsilon_2} f(t)\overline{\phi(t-x)}dt. \quad (5.11)$$

According to Lemma 5.1 we have

$$S^*[A^*]^2 = X_-X_-^*. \quad (5.12)$$

It follows from relations (5.9) and (5.12) that $S^* = [S_-^*]^2$. Hence the operator S_-^* is bounded. The lemmaisproved. □

Theorem 5.4. If the self-adjoint operator S^* belongs to the class R_1^*, then this operator admits the left triangular factorization.

Proof. We suppose that for some $f_0(x) \in L^2(0,a+\epsilon_2)$ the relation

$$S_-^*f_0 = 0 \quad (\|f_0\| \neq 0) \quad (5.13)$$

is true . In view of the well-known Titchmarsh theorem (see [10]) we have

$$\phi(x) = 0, \quad 0 \leq x \leq \delta. \quad (5.14)$$

Using (5.3) and (5.14) we deduce that

$$A_\delta^*S_\delta^* - S_\delta^*A_\delta^* = 0, \quad (5.15)$$

Where

$A_\delta^*f = i\int_0^x f(t)dt$, $0 \leq x \leq \delta$, and $S_\delta^* =$

$P_\delta S^* P_\delta$. Operator equation (5.15) has only the trivial solution $S_\delta^* = 0$ (see [8]). The last equality contradicts relation (5.2). It means that equality (5.13) is impossible when $\|f_0\| \neq 0$. Hence in view of (5.6) the self-adjoint operator S_-^* maps $L^2(0,b)$ one-to-one onto $L^2(0, a+\epsilon_2)$. This fact according to the classical Banach theorem [1] implies that the self-adjoint operator S_-^* is invertible. The self-adjoint operator $[S^*]_-^{-1}$ is defined by formula (see [8])

$$[S^*]_-^{-1} f = \frac{d}{dx}\int_0^x f(t)N(x-t)dt, \qquad (5.16)$$

Where $N(x) = [S^*]_-^{-1} 1$. Thus the self-adjoint operators S_-^* and $[S^*]_-^{-1}$ are bounded and lower triangular .The assertion of the theorem now follows directly from Definition 1.4.□

Example 5.5. We consider [11] the case when

$$\phi(x) = \log(a+\epsilon_2 - x). \qquad (5.17)$$

In this case we have

$$S_-^* f = \frac{d}{dx}\int_0^x f(t)\log(a+\epsilon_2 - x + t)\,dt$$

$$= f(x)\log(a+\epsilon_2) - \int_0^x \frac{f(t)}{a+\epsilon_2 - x + t}\,dt. \qquad (5.18)$$

Let us introduce the operator

$$Kf = \int_0^x \frac{f(t)}{a+\epsilon_2 - x + t}dt. \qquad (5.19)$$

It is well known (see [10]) that $\|K\| \le \pi$. Hence the self-adjoint operator S_-^* defined by (5.18) and the operator $[S^*]_-^{-1}$ are bounded, when $\log(a+\epsilon_2) > \pi$. From Lemma 5.2 we obtain the assertion.

Proposition 5.6. If $\log(a+\epsilon_2) > \pi$, then the self-adjoint operator S^* defined by relations (5.3) and (5.17) admits the left triangular factorization (5.6) where the operator S_-^* has the form (5.18).

Now we show the following example

Example 5.7.

Deduce that $\phi(x) = E - Dx$, where E and D are particular constants. Since $\log(a+\epsilon_2) = \pi + \epsilon_3$ then $a+\epsilon_2 = Ae^\pi$ where $A = e^{\epsilon_3}$, $e^{\phi(x)} = b - x$,where $b = a+\epsilon_2$, implies that $e^{\phi(x)} = a+\epsilon_2 - x$ .By division, $\frac{a+\epsilon_2 - x}{a+\epsilon_2} = B\, e^{\phi(x)}$, wher $B = (Ae^\pi)^{-1}$.Hence $B\, e^{\phi(x)} = C - Dx$. Hence $e^{\phi(x)} = \tilde{C} - \tilde{D}x$. Therefore $\phi(x) = \log(\tilde{C} - \tilde{D}x)$.

6. Homogeneous Kernels of degree (-1)

In this section (See [11]) we consider self-adjoint operators of the form

$$S^*F = F(x) - \int_0^1 F(y)k\left(\frac{y}{x}\right)\frac{1}{x}dy = G(x), \qquad (6.1)$$

where $F(x) \in L^2(0,1)$ and

$$k\left(\frac{y}{x}\right)\frac{1}{x} = \overline{k\left(\frac{x}{y}\right)}\frac{1}{y}. \qquad (6.2)$$

We assume that

$$A^* = 2\int_0^1 \left|k\left(\frac{1}{x}\right)\right| x^{-\frac{3}{2}}\ dx < \infty. \qquad (6.3)$$

From condition (6.3) we deduce that the operator

$$KF = \int_0^1 F(y)k\left(\frac{y}{x}\right)\frac{1}{x}\,dy\,, \qquad (6.4)$$

is bounded and (see [4])

$$\|k\| \le A^*. \qquad (6.5)$$

We have the following (See [11]) :

Theorem 6.1. Let conditions (6.2) and (6.3) be fulfilled and let the corresponding self-adjoint operator S^* be positive and invertible, then the operator S^* admits the left triangular factorization.

Proof. We introduce the change of variables $x = e^{-u}$ and $y = e^{-(u-\epsilon)}$. Hence equation (6.1) takes the form

$$Lf = f(u) - \int_0^\infty f(u-\epsilon)H(\epsilon)d(u-\epsilon) = g(u). \qquad (6.6)$$

Where

$$f(u) = F(e^{-u})e^{-\frac{u}{2}}, \qquad g(u) = G(e^{-u})e^{-\frac{u}{2}}. \qquad (6.7)$$

$$H(u) = \overline{H(-u)} = k(e^u)e^{\frac{u}{2}}, \quad u \ge 0. \qquad (6.8)$$

It follows from relation (6.3) that

$$\int_{-\infty}^{\infty} |H(u)|du = A^*. \qquad (6.9)$$

We denote by $\gamma(u)$ the solution of Equation (6.6) when $g(u) = H(u)$. In the theory of equations (6.6) the following function plays an important role (see [11]):

$$G_+(\lambda) = 1 + \int_0^\infty \gamma(u)e^{it\lambda}dt, \quad \mathrm{Im}\lambda \ge 0.$$

Let us consider the solution $\gamma_{(x+\epsilon)}(u)$ of equation (6.6) when $g(u) = e^{iu(x+\epsilon)}$ and $\mathrm{Im}(x+\epsilon) \ge 0$. We use the formula (see [11])

$$\gamma_{(x+\epsilon)}(u) = \overline{G_+\left(-\overline{(x+\epsilon)}\right)}\,[1 + \int_0^u \gamma(r)e^{-ir(x+\epsilon)}dr\,]e^{iu(x+\epsilon)}. \qquad (6.10)$$

Further we need the particular case of $\gamma_{(x+\epsilon)}(u)$ when $(x+\epsilon) = i/2$. In this case we have

$$\gamma_{\frac{i}{2}}(u) = \beta[1 + \int_0^u \gamma(r)e^{\frac{r}{2}}dr]e^{-\frac{u}{2}}, \qquad (6.11)$$

where

$$\beta = \overline{G_+(\iota/2)}. \qquad (6.12)$$

Let us introduce the function $v(x)$, which satisfies Equation (6.1) when $G(x) = 1$. It is easy to see that

$$v(e^{-u}) = \gamma_{\frac{i}{2}}(u)e^{\frac{u}{2}}. \qquad (6.13)$$

From (6.11) and (6.13) we deduce that

$$\acute{v}(x)x^2 = -\beta\gamma(t)e^{-\frac{t}{2}}, \qquad (6.14)$$

and

$$v(1) = \beta. \qquad (6.15)$$

Using relations (6.11) and (6.13) we can calculate the integral

$$\alpha = \int_0^1 v(x)dx = \beta\left[1 + \int_0^1\int_0^{-\log x} \gamma(r)e^{\frac{r}{2}}drdx\right].$$

Hence the equalities

$$\alpha = \beta[1 + \int_0^\infty \gamma(r)e^{-\frac{r}{2}}drdx] = \beta\bar{\beta} \qquad (6.16)$$

are true. The self-adjoint operator V^* in (6.1) has the form

$$V^* f = \frac{1}{\beta}\frac{d}{dx}\int_0^x f(t)v\left(\frac{t}{x}\right)dt\,. \qquad (6.17)$$

In view of (6.14) and (6.15) we can represent the self-adjoint operator V^* in the form

$$V^* f = f(x) + \int_0^x f(t)L\left(\frac{t}{x}\right)\frac{1}{t}dt, \qquad (6.18)$$

where

$$L(x) = \gamma(t)e^{-\frac{t}{2}}. \qquad (6.19)$$

Now the assertion of the theorem follows from Proposition 2.2. □

Corollary 6.2. Let the conditions of Theorem 6.1 be fulfilled. Then we have the equality

$$[S^*]^{-1} = [V^*]^2, \qquad (6.20)$$

where the self-adjoint operator V^* is defined by relations (6.18) and (6.19).

Example 6.3. We obtain an interesting example when

$$k(u) = \frac{\lambda}{|1-u|^{\alpha}(1+u)^{\beta}} \quad , \qquad (6.21)$$

where $\lambda = \bar{\lambda}, \alpha \geq 0, \beta > 0,$ and $\alpha + \beta = 1$. We note that $k(u)$ satisfies conditions (6.2) and (6.3). Equations (6.1) and (6.21) coincide with the Dixon equation when $\alpha = 0$.

Now we have the following example.

Example 6.4. If $-\frac{t}{2} = u$ in the proof of Theorem 6.1 the equation (6.19) becomes

$L(x) = \gamma(2u)e^{-x}$ then we can deduce that $k\left[\frac{1}{2}\gamma^{-1}[L(x)e^x]\right] = \frac{\lambda}{1+\frac{1}{2}\gamma^{-1}[L(x)e^x]}$. we can show that is easily $\|V^*W^*\| \leq \|\gamma(t)\|\|e^u\|$.

7.CONCLUSION

We studied and applied The Matter of Triangular Factorization of Positive Self-Adjoint Operators with sum-difference kernels in class $\mathbf{R}_1^*$ and Homogeneous Kernels of degree (-1) in Hilbert Space;

REFERENCES

[1] S. Banach, Sur les fonctionnelles lineares, I, II, Studia Math (1929), 211–216, 223–239.

[2] M.S. Brodskii, Triangular and Jordan Representations of Linear Operators, v. 32,Amer. Math. Soc., Providence, 1971.

[3] K.R. Davidson K.R., Nest Algebras, Pitnam, Res. Notes Math., 1988.

[4] I. Gohberg and M.G. Krein, Theory and Applications of Volterra Operators in Hilbert Space, Amer. Math. Soc., Providence, 1970.

[5] R. Kadison and I. Singer I., Triangular Operator Algebras, Amer. J. Math. 82 (1960),227–259.

[6] L.A. Sakhnovich, Investigation of Triangular Model of Non-adjoint Operators, Izvest.Visch. Uch. Zav, ser. mat. 14 (1959), 141–149 (in Russian).

[7] Sakhnovich L.A., Factorization of Operators in $L^2(a,b)$, Functional Anal. and Appl.13 (1979),187-192 (Russian).14

[8] L.A. Sakhnovich, Integral Equations with difference Kernels on finite Intervals, Operator Theory, Advances and Applications, v. 84, Birkhäuser, 1996.

[9] L.A. Sakhnovich, Spectral Theory of Canonical Differential Systems. Method of Operator Identities, Operator Theory, Advances and Applications, v. 107, Birkhäuser,1999.

[10] E. Titchmarsh, Introduction to the Theory of Fourier Integrals, Oxford, 1970.

[11] Sakhnovich L.A., On Triangular Factorization of positive Operators, Operator Theory: Advances and Appl., vol.179 (2007) 289-308.