The Problem With Quality And Quantity Dependent Demand

N. Konda Reddy, K. Pavan Kumar, AVN Murthy

Abstract: Today's manufacturing industries are facing lot of threat and competition and problems in meeting challenges of changing market conditions. Inventory is the heart of any organizational body which needs lot of focus in planning of procurement if inventories from different suppliers and vendor and need very good storage conditions before they are issued to the production process. The level of inventory balance is necessary taking demand and supply of the finished goods in the market by big corporate and end users. The cost of inventory has significant effect in determining the cost price of the output. The cost of procurement, cost of holding, and storage cost are very much sensitive to each other in defining the success story of the industries. The experts in the field of cost accountancy have suggested various models of cost calculation so that the industries can continuously undertake the production process without nay fluctuations and variations.

key words: Inventory, Quantity, Quality, EOQ, Stock Dependent Demand.

1. INTRODUCTION

THE NATURE OF INVENTORY PROBLEMS

Inventory is the basis of any organizational survival. Inventory is said to be the core component of any production activity which has strong economic value. Inventory refers to raw material, work in progress, consumables and non consumables, spares etc which has a shelf value and use value. Inventory enables the production activity to continuously provide inputs to the process of manufacturing in industries. Inventory procurement and storage is given at most importance in manufacturing industries. In manufacturing industries comes to an halt if there is a problem with the regular supply and storage of inventories.

THE BEHAVIOR OF DEMAND, SUPPLY AND THEIR MATHEMATICAL TREATMENT

The behavior of inventory supply depends upon demand and supply of finished goods in the market both the factors are too sides of the same coin which has different parameters for procurement and storage. Procurement directly relates to the negotiation of terms and conditions of external resources which determines negotiation of price, quantity and quality. Whereas the internal activities relate to life of the inventory, storage conditions of the inventory and method of issue of inventory to various activities. The inventory level of a commodity at time t is denoted by I(t) and it is a fundamental variable for the mathematical study of the inventory system. Let I(0) be the initial inventory, S(u) and D(u) denote respectively the supply and demand at any time u, $0 \le u \le t$. Then I(t) can be expressed as

$$I(t) = I(0) + \int_0^t \{S(u) - D(u)\}\,du$$

I(t) could be positive or negative and for a known T, the inventory held during (0,T) becomes

$$\int_0^T I(t)\,dt$$

Negative inventory is simply a shortage.

The study of I(t) and its behaviour accounts for proper modeling of inventory costs.

VARIOUS COSTS AFFECTING INVENTORY CONTROL

There are various costs associated with inventory maintenance. The main among them are as follows.

Holding Cost (h)

The success of inventory depends on two factors carrying cost and holding cost. It is being suggested and advised by the experts in the industries that all the inventories must have a strong turnover of usage value which will reduce the burden of holding and other related costs, which may lead to cost control. So big manufacturing industries always try to follow different techniques of issue of materials so that cost of holding is not a burden while determining the output level at a non fluctuating price. The common method of modeling inventory holding cost is to assume that it is proportional to the average inventory held during the period. Let T denote the planning period during which the inventory is held. Then the average inventory during a period (0,T) is given by

$$\bar{I} = \frac{\left[\int_0^T I(t)\,dt\right]}{T}$$

The average holding cost during the period is simply $h\bar{I}$. If CP denote cost price of the item excluding the cost of capital, then the holding cost per unit can be written as H = iCP, where i denotes the rate of interest or cost of capital.

Shortage Cost / Stock-out Cost (π)

Normally industries always try to determine the desirable level of inventory which can be accommodated and stored for a specific period of time. It is expected that industries desired to have continues cycle of production so that inventory will not lie idle for a longer period of time. Because of market changing conditions of prices of material the industries try to balance between minimum and maximum stock level of each stock so that there is no under pricing and over pricing of the material.

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Set up cost/Ordering cost/Replenishment cost (A) The procurement cost or ordering cost has an impact on the long and regular process of material buying from different vendors and suppliers. Since procurement cost is fixed in nature there has to be some rationality in frequency of buying inventories. In some cases the ordering cost could also include a variable component like A = A′ + bQ, where b is a positive constant that represents a component like material handling charges (loading / unloading). A′ is a fixed component independent of the lot size.

Material Cost

Material is the prime ingredient of production process and determination production cost. Different production activities need different types of materials to manufacture different types of goods. Many techniques are being followed while issuing the material at different time to the production units. That total quantity is divided into lots and unit cost are calculated to estimate the cost of raw material in the cost sheet by the accounts departments.

Operating Costs / System Control Costs

These are costs associated with data processing, forecasting, and stock reviewing and placing timely orders, check up the material flow in the system etc., which are essential for implementing any inventory policy.

THE EOQ FORMULA AND ITS ROLE IN INVENTORY MODELING

The earliest formulation of a mathematical model for the deterministic inventory problem was due to Wilson (1929) who had proposed the idea of Economic Order Quantity (EOQ). This model has created sufficient interest among researchers. His model is based on the following assumptions.

- Demand is deterministic at a constant rate of D units per unit of time.
- The rate of replenishment is infinite and there is no lead time.
- Shortages are not allowed.

Basing on the above assumptions, the sum of the costs per unit time given by

$$K(Q) = \frac{AD}{Q} + \frac{hQ}{2}$$

This model is based on the linear form of I(t) and the optimal value of Q is determined by minimizing K(Q) with respect to Q. This gives the square root formula $Q^* = [2AD/h]^{1/2}$, which is known as the Wilson's Economic Order Quantity and the minimum cost of operating the policy is $[2ADh]^{1/2}$ plus the cost of the material. In many cases, the researcher verifies whether an improved model reduces to (1.6.1) when the additional features are deleted. It is a convex function and hence a unique minimum exists. The above formula is based on the assumption that shortages are not allowed. If they are allowed and treated as backlogged at a cost of π per unit, then the optimal lot size can be shown to be

$$Q^* = \sqrt{\frac{2AD(\pi + h)}{h\,\pi}} \qquad (1)$$

and the optimal back order level is

$$Z^* = \frac{hQ^*}{(h + \pi)} \qquad (2)$$

It means that there is an optimal combination of positive and negative inventories in the EOQ system. Formulae (1) and (2) are treated as expressions in the study of inventory system. Sometimes the demand rate is not known exactly but can be described by a random variable X with probability density function (pdf) f(x) during a fixed interval of length T. The problem is to determine the order size. Let the beginning of stock for the period denoted by S, which is known as the order level. The optimal value of S is obtained by minimizing sum of expected holding and shortage costs, given by the function

$$E\{K(S)\} = h\int_0^S (S - X) f(x)\,dx + \pi\int_S^\infty (X - S) f(x)\,dx$$

This is one form of model for the classical newspaper boy problem and it is based on the assumption that the demand is instantaneous and not uniform during the period. On the other hand, if the demand occurs uniformly during the period, then the holding and shortage costs will be time-dependent. In this case, the optimal S denoted by S^* is the solution of the equation

$$\int_0^{S^*} f(x)\,dx + S^*\int_{S^*}^\infty \frac{f(x)}{x}\,dx = \frac{\pi}{\pi + h}$$

POWER PATTERN DEMAND AND ITS NATURE

The classical EOQ model assumes that the demand occurs at a constant rate throughout the inventory cycle. With this assumption the on-hand inventory at any time t becomes I(t) = Q – Dt, $0 \le t \le T$ which is a linear function of t. This assumption is not always true and the demand may behave in a peculiar way. For instance, a major portion of the demand may occur at the beginning of the cycle and slowly gets settled at a constant level, as time elapses. It is also possible that the demand rate may be low at the beginning and high at the end of the period. Such a phenomenon creates a non-linear pattern for the on-hand inventory. Let T be the fixed length of cycle, S be the beginning stock and D be the constant rate of demand. The general class of patterns for the on-hand inventory can be represented by the following function indexed by a parameter 'n'.

$$I(t) = S - D\left(\frac{t}{T}\right)^{\frac{1}{n}}, \; 0 \le t \le T$$

such that I(0) = S and I(t) = S – D. When n = 1, I(t) becomes $\left\{S - D\frac{t}{T}\right\}$ which is the linear pattern; when n > 1, a major portion of the demand occurs at the end of the period. Two extreme cases with n = 0 and n = ∞ represent the instantaneous demand occurring at the beginning or at the end of the period.

STOCK DEPENDENT DEMAND

In the classical inventory models it is assumed that the demand rate is constant and not influenced by the amount of stock maintained. Sometimes the demand is influenced by certain factors, which are briefly described below. Marketing researchers often observe that for certain items, the quantity displayed can have a motivational affect on the sales. According to Gerchack and Wang (1994), an increase in the shelf space for an item, is known to induce more customers to buy it, perhaps with the belief that the item is popular. Similarly a low stock of baked foods on the shelf, leads to an impression that the item is possibly not fresh. Silver and Peterson (1985) have observed that for many consumer items at a retail level, the sales tend to be proportional to the inventory displayed. There are two different forms of modeling SDD. The first one is based on the lot size received and displayed at the showroom. The demand is then characterised by the model $D(Q) = D + \eta Q$, where D denotes the stable portion of demand, ηindicates the linear effect of stock and Q denotes the lot sizeA basically different form of SDD is based on the on-hand inventory I(t) at time 't' given by $D[I(t)] = D + \eta I(t)$. A multiplicative form of the type $D[I(t)] = D[I(t)]^{\gamma}$ is also used by some researchers. This may be called Inventory Level Dependent (ILD) Demand. The effect of SDD cannot exist throughout the inventory cycle. The motivational effect of stock vanishes after a time lag, say a weekdays after the arrival of the stock.

2. PROBLEM ENVIRONMENT

We consider a situation in which the demand for the product depends both on the quantity ordered (and received) and the quality of the items in stock. The motivation for this type of problem has arisen out of some consumables items in which the consumer arrives at a judgment on the quality of the stock and decides to buy or not. The buying behaviour of the consumer is some times influenced by both (i) the stock on display and (ii) the quality of the items. When the demand rate is influenced by the stock on hand, it is often called the Stock-Dependent Demand. One way of modeling this type of demand is to use a function of the type $D(Q) = D+ \eta Q$, where D denotes the constant or stable portion of demand rate and η denotes the linear effect of the lot size, on demand. This is called Lot Size Dependent Demand (LSDD). It is also a practice to describe demand as a function of the on-hand inventory I(t) and a linear model of the form $D(t) = D + \eta I(t)$ can be used. This is called Inventory Level Dependent Demand (ILDD). We however restrict our focus to the LSDD and understand SDD in this sense only. The impact of product quality on the demand can be explained in terms of a quality characteristic that behaves as a random variable with a known distribution having mean μ and standard deviation σ. We assume that the supplier's quality history is stable with periodic shifts in the mean quality level μ, which means that σ is a known constant. We use a special type of demand function, which is stock dependent as well as conditioned on the quality level of the stock. We denote the quality status of the item by a parameter φ defined as $\varphi = (\mu_1 - \mu_o)$ where μ_o denotes the required quality level and μ_1 is the available quality level. Suppose the quality is normal as expected by both the stockist and the supplier. Then $\mu_1 = \mu_o$ so that $\varphi = 0$. When there is a positive shift (hike in quality) in the quality level, then φ tends to be >0 and if there is a negative shift (fall in quality) then φ will be< 0. The next problem is to estimate the value of φ. Though this value is measurable with the help of a sample, we cannot expect the customer to do this every time before making a purchase. So an indexing value based on an attribute type judgment can be attached to φ. With the available quality, if many customers show enthusiasm in buying, we may assign a value like $\varphi = 0.5$ or a value between 0 and 1. If the customers show poor interest we may attribute $\varphi = -0.5$ or a value between -1 and 0. If the customers are indifferent towards the quality, we may assign $\varphi = 0$. We however restrict φ from taking extreme values like –1 and +1. An enterprising stockist can estimate this factor using Delphi method. A scientific study on the consumer's behaviour would also help in estimating the value of φ. Both fall and hike in quality are assumed to be in relation to the expectations of the customers. Since the behaviour differs from customer to customer, a randomly chosen customer may either accept or reject the item with some positive probability. Now we use demand function of the type

$$D(Q, t) = D + \eta Q + \varphi D e^{-\varphi t}, \quad t >= 0 \qquad (3)$$

With this type of demand, we wish to determine the EOQ for a simple deterministic situation. In the following section we discuss the behaviour of the demand and the on-hand inventory situation as a function of time, quality level and the lot size,

3. BEHAVIOUR OF DEMAND

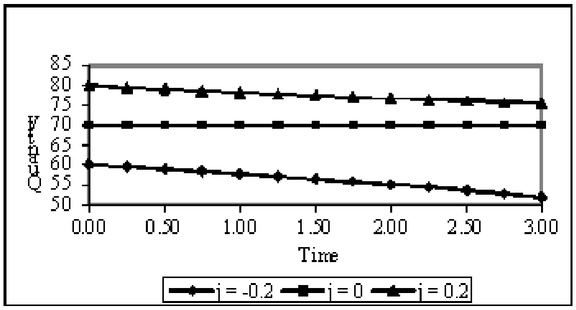
At different time points, the demand generated by (3) has been evaluated taking Q = 200, D = 50, $\eta = 0.1$. The results are shown in the table: 1 for different values of φ.

Table-1: *Quantity demanded at different time points*

t	*φ = - 0.2*	*φ = 0*	*φ = 0.2*
0.00	60.00	70.00	80.00
0.25	59.49	70.00	79.51
0.50	58.95	70.00	79.05
0.75	58.38	70.00	78.61
1.00	57.79	70.00	78.19
1.25	57.16	70.00	77.79
1.50	56.50	70.00	77.41
1.75	55.81	70.00	77.05
2.00	55.08	70.00	76.70
2.25	54.32	70.00	76.38
2.50	53.51	70.00	76.07
2.75	52.67	70.00	75.77
3.00	51.78	70.00	75.49

When $\varphi = 0$ we see no change in D(t) whereas at $\varphi \neq 0$ we notice a fall in D(t). This is due to the third term in (3). When $\varphi = -0.2$ a status of poor quality is indicated. As time passes, the customers get a spread of knowledge about the quality and hence the demand rate shows a decline. If $\varphi = 0.2$ there is a clear indication of better quality than what is expected by the customer. So the tendency of the customer will be to purchase the items in the early part of the inventory cycle. This is something like a 'hasty purchase'. But as time passes the same tendency would not work and the demand arte slowly declines. This behaviour is shown in figure-1.

Figure-1: *Behaviour of demand during the cycle*



In the following section we examine the behaviour of the on-hand inventory with this type of function.

BEHAVIOUR OF THE ON-HAND INVENTORY

Let I(t) denote the inventory on hand at any time during the cycle. Then the system dynamics can be described by the differential equation

$$I(t+\delta t) = I(t) - D - \eta Q - \varphi De^{-\varphi t}$$

=> $I'(t) = -D - \eta Q - \varphi De^{-\phi t}$, for $0 \le t \le t_0$, where t_0 is the time at which the inventory drops to zero.

The solution of this equation, using the initial condition I(0) = Q gives

$I(t) = De^{-\varphi t} - Dt - \eta Qt + (Q - D)$, $t \ge 0$ (4)

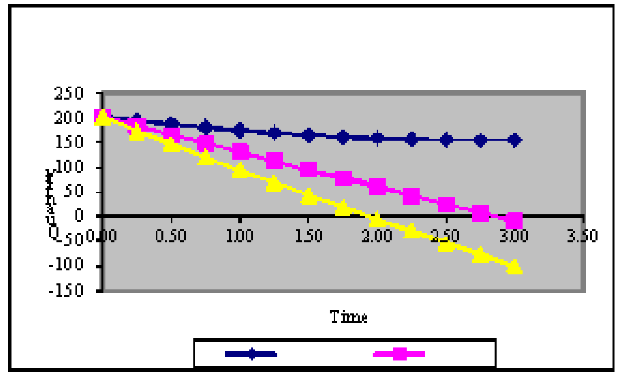
The behaviour of I(t) for different values of t and γ with Q = 200 and η = 0.1 is shown in Table-2.

Table-2: *On hand inventory at different time points*

t	*φ = - 0.2*	*φ = 0*	*φ = 0.2*
0.00	*200.00*	*200.00*	*200.00*
0.25	*192.75*	*182.50*	*172.75*
0.50	*186.03*	*165.00*	*145.97*
0.75	*179.87*	*147.50*	*119.64*
1.00	*174.28*	*130.00*	*93.75*
1.25	*169.31*	*112.50*	*68.26*
1.50	*164.97*	*95.00*	*43.16*
1.75	*161.31*	*77.50*	*18.44*
2.00	*158.36*	*60.00*	*-5.94*
2.25	*156.16*	*42.50*	*-29.97*
2.50	*154.74*	*25.00*	*-53.69*
2.75	*154.15*	*7.50*	*-77.11*
3.00	*154.42*	*-10.00*	*-100.24*

As a consequence of the demand function used in this model, the on-hand inventory takes a marginally non-linear form during the cycle. When φ = 0.2, the on-hand stock has become negative showing shortages. If shortages are not allowed then the re-order takes place whenever I(t) becomes zero. The behaviour of I(t) is shown in Figure-2.

Figure-2: *Behaviour of on-hand inventory during the cycle*



THE LOT SIZE MODEL WITH SDD AND QDD

Consider the EOQ situation in which shortages are not allowed. An order for Q units is placed whenever the stock level comes to zero. The order is received without any lead time and the entire lot is delivered in a single replenishment. The cost of ordering is A, the cost of holding one unit per unit time is h and the unit selling price is p per unit while the purchase cost of each unit is c. Thus (p-c) becomes the profit per unit.

From (4) we can find the time t_0 at which I(t) falls to zero by solving the equation $I(t_0) = 0$. This gives $t_0 (D + \eta Q) = De^{-\varphi t_0}$ and the inventory held during the cycle is given by $\int_0^{t_0} I(t)dt$. Since t_0 has no explicit form in Q, it is difficult to work out the holding cost in this form. sing Taylor's approximation to the exponential functions and ignoring the second and higher order terms, I(t) can be approximated as, $I(t) \cong (Q - Dt) - (\eta + \varphi)t$. and t_0 becomes

$$t_0 \cong \frac{Q}{(D+\eta+\varphi D)} \text{ or } \frac{Q}{D(1+\varphi)+\eta Q}$$

When η and φ become zero the value of t_0 reduces to Q/D, which is the cycle length in the classical inventory model.

The holding cost during the cycle is given by

$$h\int_0^{t_0}[Q - Dt\ \eta\ Qt - D\varphi t\ ]dt$$

$$= h\left[Qt_0 - \frac{t_0^2}{2}[D+\eta Q+D\varphi]\right]$$

The sum of costs in each cycle therefore becomes

$$\hat{K}(Q,t) = A + cQ + h\left\{Qt_0 - \frac{t_0^2}{2}[D+\eta Q+D\varphi]\right\}$$

The normal duration of the inventory cycle is Q/D when there are no additional influencing factors like SDD or QDD. If Q units are held in stock, then $(Q - Dt_0) = 0$. Given the stable portion of the demand rate, we expect the stock to be sold out by time Q/D. But due to the external factors affecting the demand, the stock gets emptied earlier than Q/D so that $(Q-Dt_0)$ will be positive.

Proposition: The additional sales due to stock and quality dependent demand is $p(Q-Dt_0)$ in each cycle.
Proof: Under the influence of SDD and QDD the cycle becomes shorter than the usual one, which means that new customers are motivated by the displayed stock and its quality. The difference $(Q-Dt_0)$ represents this extra sales and it fetches a revenue to the tune of $p(Q-Dt_0)$. When $\eta = 0$ and $\varphi = 0$ this component vanishes.
Hence the proof.
The net inventory cost in each cycle is obtained as the sum of ordering, holding and material costs, minus the revenue from additional sales. This is given by

$$K(Q,t) = A + cQ + h\left\{Qt_0 - \frac{t_0^2}{2}\left[D + \eta Q + D\varphi\right]\right\} - p\left(Q - Dt_0\right)$$

and the net cost per unit time becomes

$$K(Q) = \frac{K(Q,t)}{t_0} =$$

$$\frac{AD}{Q}(1+\varphi) + Q\left(c\eta + \frac{h}{2} - p\eta\right) + \left(A\eta + cD + cD\varphi - pD\varphi\right)$$

.
Due to the similarity of this cost function with that of the simple EOQ model, K(Q) is convex in Q and the value of Q that minimizes K(Q) is the solution of $\frac{dK(Q)}{dQ} = 0$ and is given by

$$Q^* = \sqrt{\frac{2AD(1+\varphi)}{h - 2\eta(p-c)}} \qquad (5)$$

Particular cases:

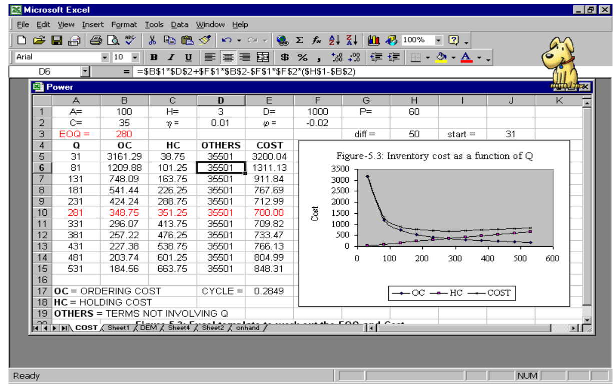
1. When $\varphi = 0$, (5) reduces to $Q^* = \sqrt{\frac{2AD}{h - 2\eta(p-c)}}$ which is the EOQ given by Prasad (1996), after claiming a correction in the Rakesh Gupta and Prem Vrat (1986) in their basic model with SDD.
2. When $\eta = 0$, (5) reduces to $Q^* = \sqrt{\frac{2AD(1+\varphi)}{h}}$ which reflects the linear effect of quality on normal demand. This formula explains how the regular demand is influenced by the quality factor even though SDD factor is ignored.

4. SENSITIVITY ANALYSIS OF THE EOQ

The behaviour of the EOQ and the corresponding cost function are studied in detail with the following input parameters.

Illustration-1:
Consider the parameters (all in consistent units) A = 100, h = 3, D = 1000, c = 35, p = 60, $\eta = 0.01$ and $\varphi = -0.02$. With these values the EOQ is found to be 280 units and the cycle length is 0.28 time units. The calculations are performed using the Excel template shown in figure-3 developed for this purpose. After evaluating the EOQ, this template automatically takes the range of values of Q around the EOQ and finds out the corresponding cost. We can change any of the parameters and find the corresponding cost. The calculations are performed using the Excel template shown in figure-3.

Figure-3: *Excel template to study the behaviour of cost function*



For different values of φ and η the following results have been obtained.

Table-3: *Sensitivity of the model to changes in φ and η*

η	φ = -0.02		
	EOQ	COST	t_0
0.00	256	766.81	0.2608
0.01	281	709.00	0.2849
0.02	314	626.10	0.3174
	φ = 0.00		
η	EOQ	COST	t_0
0.00	258	774.60	0.2682
0.01	283	707.11	0.2820
0.02	317	632.46	0.3142
	φ = 0.02		
η	EOQ	COST	t_0
0.00	261	782.30	0.2557
0.01	286	714.14	0.2793
0.02	320	638.75	0.3112

The above values show that the EOQ as well as the cost changes significantly with an increase in φ. The value of $\varphi = -0.02$ indicates that the customers assessment on the quality is on the negative side which means the demand rate will not be as it was when $\varphi = 0$ (perfect quality). We however order for a higher quantity of 283 units (not 258) as in the case of $\eta = 0$. This is the effect of SDD on the inventory policy.

5 CONCLUSION

In this a simple mathematical model has been developed to combine the effect of both quality and quantity, on the demand for the product. This is a basic version of a general class of models that can be used to explain demand. For the particular set of parameters choosen in the illustration, the effects are however not dominantly visible on the optimal solution. This model can be strengthened by considering non-linear patterns also.

Conflict of Interest: The authors declare that there is no conflict of interest in this publication.

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