Scattering of Capillary Waves In Front of A Semi-Infinite Dock In An Ocean With Porous Undulatory Bottom

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Abstract:- The problem of scattering of capillary waves by a semi-infinite dock with a porous undulatory sea bed is attempted using perturbation technique. Two different eigen function expansions of the velocity potential function on the two sides of the surface discontinuity are constructed. An integral expression for first order reflection coefficient is arrived at involving the bottom shape function. The first order reflection co-efficient is graphically depicted in case of a sinusoidal bottom topography for various values of the porous parameter and selected values of the surface tension parameter.

Keywords:- bottom undulation, capillary waves, eigen function expansion, perturbation technique, porous sea bed, reflection coefficient, semi-infinite dock, surface discontinuity, surface tension.

1 INTRODUCTION

Problems involving scattering of water waves due to a discontinuity in the free surface with a variable bed topography have been dealt with by many authors working in the field of fluid dynamics and ocean engineering. The free surface discontinuity arises, as an instance, when half the surface of water is free and the remaining half is covered by a dock extended upto infinity. The dock problem was formulated mathematically by Friedrich and Lewy [1]. Chung and Linton [2] found analytic expressions of hydrodynamic coefficients relating to scattering of waves across a finite gap between two semi-infinite elastic plates using techniques of residue calculus. Linton [3] made use of modified residue calculus technique to study the scattering problem in presence of a finite dock. Chakraborti, Mandal and Gayen [4] revisited the semi-infinite dock problem using Fourier analysis and singular integral equation. In this context, problems concerning scattering of wave due to a variable bottom geometry are also relevant. In this connection Mandal and Basu [5] studied the water wave scattering problem in presence of bottom undulation and surface tension in the free surface using perturbation expansion in terms of bed undulation parameter. Mandal and De [6] investigated surface wave propagation over small undulation at the bottom of an ocean with surface discontinuity using an eigen function expansion method. These problems are, in general, difficult to solve analytically although there exist several approximate and numerical methods to find expressions of physical interest such as reflection and transmission coefficients. Roseau [7] obtained an explicit analytic solution for the two-dimensional problem of wave propagation over a particular bottom topography.

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Peters [8], Weitz and Keller [9] considered the Wiener-Hopf technique to study the propagation of surface waves at an inertial surface composed of a thin but uniform distribution of non-interacting floating materials, e.g. broken ice, floating mat, etc. on one side and the free surface on the other side. Clearly, the discontinuity arises due to two different types of boundary conditions on the two halves. Martha and Bora [10] applied Fourier transform to analyse scattering of surface waves by small undulation on a porous sea-bed. The present paper examines scattering of capillary waves arising out of surface tension in the free surface region in front of a semi- infinite dock in a sea with porous bottom undulation. The capillary waves as an effect of surface tension may arise, for example, due to presence of, say, oil, in front of the dock. Since the water below the dock is undisturbed there will be no transmission of incoming wave-train and hydrodynamic parameter of interest in the related scattering phenomena is the reflection coefficient. The bottom boundary condition suggest a small perturbation parameter expansion of the velocity potential and the reflection coefficient. The problem is segregated into two boundary value problems, one for the zero order and another for the first order potential function. The potential functions are expanded on either side of discontinuity in terms of orthonormal eigen functions. These eigen functions involve the physical quantities such as wave number, depth and porous parameter. Besides, they also involve unknown constants and unknown reflection coefficients. A system of linear equations involving the constants is arrived at using the matching conditions of the potential functions and its first order derivatives at the junction of discontinuity and keeping in mind the orthonormality of depth eigen functions. The zero order reflection coefficient is obtained by recollecting the corresponding expressions given by Evans and Linton [11]. Next, Green's second integral theorem is suitably applied to get the expression for the first order reflection coefficient. It has been ensured graphically that the relevant dispersion relation possesses a unique non-trivial positive real root. The first order reflection coefficient is plotted against wave number for different values of the porosity parameter and selected values of surface tension parameter in the particular case of sinusoidal bottom topography. The present work gains importance in studies on coastal dynamics and marine science where construction of dock of a specified geometry is of frequent occurrence. This paper is also relevant in connection with ongoing scientific activities in polar regions, including the investigation of scattering of water waves by an obstacle with an uneven sea-bed,

2 MATHEMATICAL FORMULATION

A two-dimensional potential flow in an ocean of finite depth with a porous bottom with small undulation is considered. A rectangular cartesian co-ordinate system with y axis vertically downwards along the depth of the ocean is chosen. The semi-infinite floating dock of small width and uniform density is assumed to occupy the other side $0 \leq x < \infty, y = 0$ of the free surface in water of finite depth h. The free surface $x \leq 0, y \leq 0$ is subject to surface tension with parameter M given by $M = T/\rho g$ where g is the acceleration due to gravity, T is the coefficient of surface tension, ρ is the density of water. The discontinuity in the surface boundary condition exists because the free surface boundary condition involving the effect of surface tension differs from the boundary condition applicable on the dock portion. A wave train with source at negative infinity is incident normally upon the semi-infinite dock and gets reflected. The direction of the real positive x axis coincides with that of the incident wave field. Since the water under the dock is undisturbed and the dock extends infinitely along the positive x - axis, any transmission of the incident wave is absent. The water in the ocean is assumed to be irrotational, incompressible, inviscid and the wave motion is time-harmonic with angular frequency ω.

Let $\psi(x, y) = Re\{\varphi(x, y)e^{-i\omega t}\}$ represent the velocity potential of the fluid motion for the two-dimensional fluid region. The mathematical problem under consideration is to solve the boundary value problem in which the potential function φ satisfies the following Laplace equation along with certain boundary conditions:

$$\nabla^2\varphi = 0 \text{ in the entire fluid region} \qquad 2.1$$

The free surface boundary condition, taking surface tension parameter M into account, is given by:

$$K\varphi + \varphi_y + M\varphi_{yyy} = 0 \text{ on } y = 0, x < 0 \qquad 2.2$$

where $K = \frac{\omega^2}{g}$, ω is the angular frequency, g is the acceleration due to gravity.

The surface boundary condition at the dock is given by:

$$\varphi_y = 0 \text{ on } y = 0, x > 0 \qquad 2.3$$

The sea-bottom boundary condition is given by:

$$\varphi_\eta - G'\varphi = 0 \text{ on } y = h + \epsilon c(x) \qquad 2.4$$

where η is the outward normal to the ocean bed and $G' = \frac{\alpha}{\sqrt{\upsilon}}$, the porous effect parameter. The quantity α is dimensionless constant with depends on the structure of the porous medium and ν is the permeability of the porous medium. Here, $y = h + \epsilon c(x)$ denotes the bottom of an ocean of variable depth, ϵ being a small non-dimensional positive number giving a measure of smallness of the ocean-bottom undulation and $c(x)$, the shape function, is a bounded function with compact support with the property that $c(x) \to 0$ as $x \to \pm\infty$. This indicates that far away from the undulation the ocean bottom is of uniform finite depth h below the mean free surface.

Let the incident wave train be described by the velocity potential $\varphi^1(x. y) = e^{ik_0x}\psi_0^1(y)$ where

$$\psi_0^1(y) = N_0^1 \frac{k_0 coshk_0(h-y) - G' sinhk_0(h-y)}{k_0 coshk_0 h - G' sinhk_0 h}$$

$$N_0^1 = \frac{2\sqrt{k_0}}{\sqrt{2k_0^3 h + 2k_0G' - 2k_0hG' - 2k_0G' cosh2k_0h + (G'^2 + k_0^2) sink_0 h}}$$

k_0 being the real root of the dispersion relation:

$$(K + G' + G'Mk^2) = \left(k + Mk^3 + \frac{KG'}{k}\right) tanhkh \qquad 2.5$$

The far field behaviour of the potential function is described by:

$$\varphi(x, y) \sim \begin{Bmatrix} 0 & as\ x \to \infty \\ \varphi^1(x, y) + R\varphi^1(-x, y) & as\ x \to -\infty \end{Bmatrix} \qquad 2.6$$

where determination of R, the unknown reflection coefficient, is our main concern.

3 METHOD OF SOLUTION

The sea-bed condition (2.4) can be approximated as:

$$\frac{\partial\varphi}{\partial y} - \epsilon\left[\frac{\partial}{\partial x}\{c(x)\varphi_x\}\right] - G'[\varphi + \epsilon c(x)\varphi_y] = 0 \text{ on } y = h \qquad 3.1$$

The scattering of the incident wave train due to free surface discontinuity and uneven ocean bottom imply that a perturbation technique can directly be applied to the governing problem. The form of (2.6) suggests that φ and R can have the following perturbation expansion in terms of variational parameter ϵ :

$$\varphi(x, y, \epsilon) = \varphi_0(x, y) + \epsilon\varphi_1(x, y) + 0(\epsilon^2) \qquad 3.2$$

$$R(\epsilon) = R_0 + \epsilon R_1 + 0(\epsilon^2)$$

We substitute the expression (3.2) in the governing PDE(2.1), the surface boundary conditions (2.2) and (2.3), the approximate bottom boundary condition (3.1) and equate the like powers of ϵ upto order one. These produce two separate boundary value problems in terms of $\varphi_0(x, y)$ and $\varphi_1(x, y)$. They are denoted by BVP-1 and BVP-2 respectively.

3A. BVP-1

The function $\varphi_0(x, y)$ satisfies

$$\nabla^2\varphi_0(x, y) = 0 \text{ in } 0 < y < h, -\infty < x < \infty \qquad 3A.1$$

$$K\varphi_0 + \varphi_{0y} + M\varphi_{0yyy} = 0 \text{ on } y = 0, x < 0 \qquad 3A.2$$

$$\varphi_{0y} = 0 \text{ on } y = 0, x > 0 \tag{3A.3}$$

$$\varphi_{0y} - G'\varphi_0 = 0 \text{ on } y = h \tag{3A.4}$$

$$\varphi_0(x,y) \sim \begin{cases} 0 \ as & x \to \infty \\ \varphi^1(x,y) + R_0\varphi^1(-x,y) \ as \ x \to -\infty \end{cases} \tag{3A.5}$$

where R_0 is the zero order reflection coefficient.

3B. BVP-2

$$\nabla^2\varphi_1(x,y) = 0 \text{ in } 0 < y < h, -\infty < x < \infty \tag{3B.1}$$

$$K\varphi_1 + \varphi_{1y} + M\varphi_{1yyy} = 0 \text{ on } y = 0, x < 0 \tag{3B.2}$$

$$\varphi_{1y} = 0 \text{ on } y = 0, x > 0 \tag{3B.3}$$

$$\varphi_{1y} - G'^{\varphi_1} = \frac{\partial}{\partial x}[c(x)\varphi_{0x}] + G'c(x)\varphi_{0y} \text{ on } y = h \tag{3B.4}$$

$$\varphi_1(x,y) \sim \begin{cases} 0 & as \ x \to \infty \\ R_1\varphi^1(-x,y) \ as \ x \to -\infty \end{cases} \tag{3B.5}$$

where R_1 is the first order reflection coefficient.

The BVP-1 corresponds to the problem of water wave scattering by a semi-infinite dock in uniform finite depth of water. This can be determined analytically using residue calculus technique of the complex variable theory which was used by Evans and Linton [11] to solve the problem of scattering of water waves in case of a discontinuity in the free surface boundary condition in an uniform depth of water. Without solving BVP-2, R_1 can be determined in terms of $\varphi_0(x,y)$ and its first order partial derivatives.

The zero order potential function $\varphi_0(x,y)$ of the BVP-1 can be expanded in terms of orthonormal sets of eigen functions in two different regions of the ocean as follows:

$$\varphi_0(x,y) = \begin{cases} \sum_{n=1}^{\infty} B_n e^{-s_n x \psi_n^2(y)} & as \ x > 0 \\ (e^{ik_0x} + R_0e^{-ik_0x})\psi_0^1(y) + \sum_{n=1}^{\infty} A_n e^{k_n x}\, \psi_n^1(y) & as \ x < 0 \end{cases} \tag{3.6}$$

where A_n, B_n are the unknown constants; k_n, s_n are the eigen values of the BVP in the free surface region $(x < 0)$ and the dock region $(x > 0)$ respectively.

The two sets of orthonormal eigen functions $\{\psi_n^1(y)\}$ and $\{\psi_n^2(y)\}$ corresponding to the free surface region $(x < 0)$ and the dock region $(x > 0)$ respectively are given by:

$$\psi_n^1(y) = N_n^1 \cdot \frac{k_n cos k_n(h-y) - G' sin k_n(h-y)}{k_n cos k_n h - G' sin k_n h}$$

And

$$\psi_n^2(y) = N_n^2 \cdot \frac{s_n cos s_n(h-y) - G' sin s_n(h-y)}{s_n cos s_n h - G' sin s_n h}$$

where,

$$N_n^1 = \frac{2\sqrt{k_n}}{\sqrt{2k_n^3 h - 2G'k_n + 2k_n h G'^2 + 2k_n G' cos 2k_n h + (k_n^2 - G'^2) sin 2k_n h}}$$

And

$$N_n^2 = \frac{2\sqrt{s_n}}{\sqrt{2s_n^3 h - 2G's_n + 2s_n h G'^2 + 2s_n G' cos 2s_n h + (s_n^2 - G'^2) sin 2s_n h}}$$

k_n, s_n being the real roots of the following equations respectively:

$$(K + G' - G'Mk_n^2) = \left(-k_n + Mk_n^3 + \frac{KG'}{k_n}\right) tan k_n h$$

And

$$s_n tan s_n h + G' = 0$$

where R_0 is the unknown reflection coefficient of zero order; $A_n, B_n, n = 1,2.....$ are the unknown constants which are to be determined. The matching conditions at $x = 0$ give:

$$\varphi(0,y)|_{x>0} = \varphi(0,y)|_{x<0}; \quad \varphi_x(0,y)|_{x>0} = \varphi_x(0,y)|_{x<0}$$

As a consequence we obtain the following relations:

$$(1 + R_0)\psi_0^1(y) + \sum_{n=1}^{\infty} A_n\psi_n^1(y) = \sum_{n=1}^{\infty} B_n\psi_n^2(y)$$

And

$$ik_0(1 - R_0)\psi_0^1(y) + \sum_{n=1}^{\infty} A_n k_n \psi_n^1(y) = -\sum_{n=1}^{\infty} B_n s_n \psi_n^2(y)$$

Using orthogonality of $\psi_m^1(y), \psi_m^2(y),\ m = 0,1,2 \ldots..$ we get:

$$\sum_{n=1}^{\infty} B_n \int_0^h \psi_n^2(y)\psi_m^1(y)dy = A_m\,; \ -\sum_{n=1}^{\infty} B_n s_n \int_0^h \psi_n^2(y)\psi_m^1(y)dy = A_m k_m$$

These imply:

$$\sum_{n=1}^{\infty} (k_m + s_n) B_n \int_0^h \psi_n^2(y)\psi_m^1(y)\, dy = 0$$

And

$$(s_m + k_n)A_n \int_0^h \psi_n^1(y)\psi_m^2(y)dy = R_0\left((ik_0 - s_m)\int_0^h \psi_0^1(y)\psi_m^2(y)dy - (ik_0 + s_m)\int_0^h \psi_0^1(y)\psi_m^2(y)dy\right)$$

where A_n and B_n can be numerically evaluated from the above system of linear equations. As estimation of R_0 as given by Evans and Linton [11] is as follows:

$R_0 = e^{i\theta}$ where $\theta = \sum_{n=1}^{\infty}\left\{tan^{-1}\left(\frac{k_0}{s_n}\right) - tan^{-1}\left(\frac{k_0}{k_n}\right)\right\}$;

Further, an application of Cauchy's residue theorem leads to the following representation of zero order reflection coefficient:

$$R_0 = \prod_{n=1}^{\infty}\frac{\left(1+\frac{k_0}{k_n}\right)\left(1-\frac{k_0}{s_n}\right)}{\left(1+\frac{k_0}{s_n}\right)\left(1-\frac{k_0}{k_n}\right)}$$

4 THE FIRST ORDER REFLECTION COEFFICIENT

The first order correction of the reflection coefficient of the incident wave field can be evaluated applying Green's integral theorem using the two potential functions $\varphi_0(y)$ and $\varphi_1(y)$ along the contour L given by:

$$L: \begin{cases} y = 0, -X \le x \le X \\ x = X, 0 \le y \le h \\ x = -X, 0 \le y \le h \\ y = h, -X \le x \le X \end{cases} \qquad 4.1$$

Along L: $\oint_L (\varphi_0\varphi_{1n} - \varphi_1\varphi_{0n})dl = 0$ 4.2

where n is the outward normal to the line element dl. The surface boundary conditions satisfied by the potential functions $\varphi_0(y)$ and $\varphi_1(y)$ ensure that there is no contribution to the integral along the path $(y = 0, -X \le x \le X)$. Again, since the water under the dock is undisturbed and there is no dispersion of incoming wave, therefore, there is no contribution to the integral along the path $(x = X, 0 \le y \le h)$. The only contribution to the integral (4.2) arises from the bottom part, i.e. along the portion $(x = -X, 0 \le y \le h)$. Finally, making $X \to \infty$ we arrive at:

$$R_1 = \frac{1}{2ik_0}\int_{-\infty}^{\infty} c(x)\left[\varphi_{0x}^2(x,h) - G'\varphi_0(x,h)\varphi_{0y}(x,h)\right]dx \qquad 4.3$$

5 A SPECIAL BED SURFACE

We consider the interaction of progressive surface waves with a patch of sinusoidal ripples on the porous sea-bed, and the ripples do not imply any restriction on the bed wave number. The bed surface is given by

$$c(x) = \begin{cases} c_0 sin(\mu x + \delta), L_1 \le x \le L_2 \\ 0 \qquad otherwise \end{cases} \qquad 5.1$$

where c_0 is the amplitude of the sinusoidal ripples. μ the wave number of the sinusoidal ripples, δ is an arbitrary phase angle. Let $L_1 = \frac{-m\pi-\delta}{\mu}, L_2 = \frac{m\pi-\delta}{\mu}$ where m is a positive integer. This represents a patch of sinusoidal ripples on an otherwise flat bottom, the patch consisting of m ripples having the same wave number μ. For such a sinusoidal sea bottom topography an explicit expression for the first order reflection coefficient is given by:

$$\begin{aligned} R_1 &= \frac{c_0}{2ik_0}\Bigg[-k_0^2(N_0^1)^2\left\{\frac{\mu}{\mu^2-4k_0^2}\left((-1)^m e^{\frac{-2ik_0 m\pi}{\mu}}-1\right)\right. \\ &\left.+\frac{\mu R_0^2}{\mu^2-4k_0^2}\left((-1)^m e^{\frac{2ik_0 m\pi}{\mu}}-1\right)+\frac{2R_0}{\mu}((-1)^m-1)\right\} \\ &+\int_{-m\pi/\mu}^{0}\left(\sum_{n=1}^{\infty}k_nA_n e^{k_n x}N_n^1\right)^2 sin\mu x dx \\ &+2ik_0N_0^1\sum_{n=1}^{\infty}\frac{k_nA_nN_n^1\mu}{\mu^2+(k_n+ik_0)^2}\left((-1)^m e^{\frac{m\pi(k_n+ik_0)}{\mu}}-1\right) \\ &-2ik_0N_0^1R_0\sum_{n=1}^{\infty}\frac{k_nA_nN_n^1\mu}{\mu^2+(k_n-ik_0)^2}\left((-1)^m e^{\frac{m\pi(k_n+ik_0)}{\mu}}-1\right) \\ &+\int_0^{m\pi/\mu}\left(\sum_{n=1}^{\infty}s_nB_ne^{-s_nx}N_n^1\right)^2 sin\mu x dx\Bigg] \end{aligned}$$

6 GRAPHS

Fig.1 indicates the existence of a unique non-trivial positive real root of the dispersion relation(2.5) ensuring the practical feasibility of the current problem.

Fig.2 and fig.3 depict the variation of the modulus of first order reflection coefficient $|R_1|$ with variation in the wave number for different values of non-dimensionalised porosity parameter $G'h$ in case of a bottom profile which varies sinusoidally.

Fig. 2 plots $|R_1|$ against Kh taking $G'h = 0.0, 0.1, 0.2$ while the surface tension parameter M is fixed at 0.09.

Fig. 3 plots $|R_1|$ against Kh taking $G'h = 0.0, 0.1, 0.2$ while the surface tension parameter M is fixed at 0.1. The graphs of $|R_1|$ are found to be oscillatory in nature for each $G'h$ and the oscillation gradually decreases with Kh. Bragg resonance is indicated in all cases where the oscillation of R_1 attains a maximum value for certain value of Kh.

Fig. 2 and fig. 3 are significantly different around the value $Kh = 1.5$. In the neighbourhood of this value, the plot of $|R_1|$ in fig. 3 attains its maximum at a higher value than the plot of $|R_1|$ in fig. 2. This is particularly pronounced in case of absence of bottom porosity $(G'h = 0)$. Thus for a sinusoidal bottom profile a slight increase in surface tension considerably changes the plot of $|R_1|$, especially when the bed is non permeable.

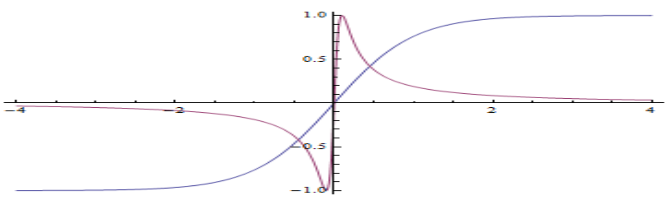

Fig. 1: Root of Dispersion Relation for G'h=0.1 and M=0.1. y=tanhxh; y=((K+G')x+MG'x^3)/(x^2+KG'+Mx^4)

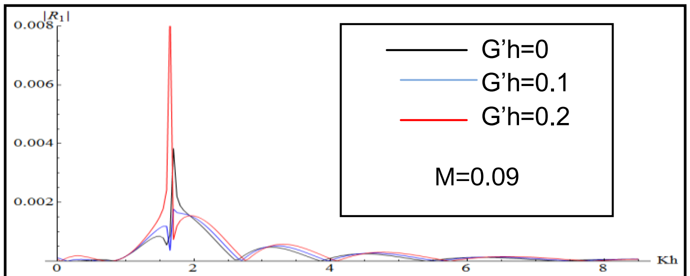

Fig. 2: Reflection Coefficient for different values of porous parameter with M=0.09

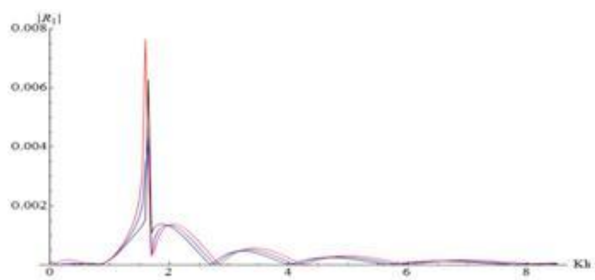

Fig. 3: Reflection Coefficient for different values of porous parameter with M=0.1

7 CONCLUSION

The zero and first order reflection coefficients for water wave scattering in front of a semi-infinite dock with porous undulatory bottom profile and surface tension in the free surface are obtained. An expression for the first order reflection coefficient is arrived at without solving the first order potential function. The graphs indicate oscillatory damping of first order reflection coefficient with increase in wave number. Furthermore, it can be concluded that presence of surface tension in the free surface and sinusoidally varying porous sea bed play their own significant roles in modifying the incoming wave train which gets reflected by the floating dock.

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