Properties And Experimental Of Gaussian And Non Gaussian Time Series Model

A. M. Monem

Abstract: Most of time series that appear in many economical geophysical and other phenomena are driven by non- Gaussian white noise (a_t), in this paper investigate some probabilistic properties of Gaussian and non- Gaussian mixed with identification methods of ARMA model. We have theoretically derived the characteristic function the first of (four moments) of skeweness and kurtosis coefficients for white noise (a_t) with Gaussian and non-Gaussian (Poisson) distribution, simulation experiments were used to confirm the accuracy of the theoretical results. Declared the identification sample Autocorrelation function (ESACF) and (Kumar) method (C- table) which depending upon the pad approximation and suggested new method depending upon the extended sample partial Autocorrelation function (ESPACF) and find ascertain efficiency of suggested method.

Index Terms: Gaussian white noise distribution, non- Gaussian white noise distribution, ARMA process, characteristic function of moments for skeweness, characteristic function of moments for kurtosis extended sample autocorrelation, extended sample partial autocorrelation

1 INTRODUCTION

Time series analysis and forecasting methods play an important role in all researchers especially with Gaussian and non Gaussian mixed models. Nelson and Granger (1979), Obeysekera and Yevjevich (1985) reported a procedure for generation of samples of an autoregressive scheme that had an Poisson distribution with given mean and skewness. Fernandez and Salas (1986) developed and tested a new class of time series models capable of reproducing the covariance structure normally found in periodic stream flow time series under non-Gaussian marginal distribution. The general class of forecasting methods involves two basic tasks, analysis of time series data and selection of the forecasting model that best fits the data series. Today, alter acrimonious arguments and considerable debates, it is accepted by a large number of researchers that in empirical tests Box-Jenkins is not an accurate method for post-sample time series forecasting, at least in the domains of business and economic applications where the level of randomness is high and where constancy of pattern, or relationships. Sim (1987) considered a time series model which can be used for simulating stationary river flow sequences with high skewness and the long-term correlation structure of an ARMA(1,1) models valid for non-normal distribution have also been developed Series of weekly stream flow were used for application and comparison of the proposed method. In this research investigate some probabilistic properties of Gaussian and non-Gaussian mixed with identification methods of ARMA (1, 1) model with derived the characteristic function of moments for Skeweness and Kurtosis coefficients for (a_t) with Gaussian and non- Gaussian (Poisson) distribution, a Simulation experiments were used to confirm the accuracy of the theoretical results.

- *M. Monem – Al Sulamania university – collage of Administration and Economic, E-mail: monem_aziz2003@yahoo.com*

2 General theoretical

Let, we have stationary time series ($Z_t = 0, \pm 1, \pm 2, \dots$), then, we have the following ARMA (P, q) process:

$$Z_t = \emptyset_1 Z_{t-1} + \emptyset_2 Z_{t-2} + \cdots + \emptyset_p Z_{t-p} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} \quad \dots (1)$$

Where $Z_t = \tilde{Z}_t - \mu$

For each (t), (μ) is the mean of time series and $\{a_t\}$is a purely random error (white noise) distributed Gaussian with E (a_t) = 0,

Variance of (a_t) = E $(a_t)^2 = \sigma_a^2$ and

Covariance ($a_t, a_{t\pm k}$) = 0, for all k≠0.

Then, ARMA (1, 1) distributed Gaussian or normal distribution. By (B) operator we get:

$$Z_t\{1- \emptyset_1 B - \emptyset_2 B^2 - \dots - \emptyset_p B^p \} = \{1- \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\}\, a_t$$

$$\therefore Z_t = \frac{\{1-\theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q\}}{\{1- \emptyset_1 B - \emptyset_2 B^2 - \dots - \emptyset_p B^p\}} a_t \quad \dots (2)$$

$\{ Z_t = \frac{\Theta_q(B)}{\Phi_p(B)} a_t\}$ Where $\Theta_q(B)$ and $\Phi_p(B)$ are the polynomial functions of order (q) and (p) in (B) respectively.

Or $Z_t = \psi(B)\, a_t$ … (3)

Where; $\psi(B) = \frac{\theta_q(B)}{\emptyset_p(B)} = \psi_0 B^0 + \psi_1 B^1 + \psi_2 B^2 + \psi_3 B^3 + \dots,\ \psi_0 = 1$

or $\psi(B) = \frac{\theta_q(B)}{\emptyset_p(B)} = \sum_{j=0}^{\infty} \psi_j B^j$, $\psi_0 = 1$, $\psi_1, \psi_2, \psi_3, \dots$, the weights

3 Mixed model with normal distribution:

Let we have a mixed model ARMA (1, 1) with normal distribution as follows:

$$(1- \emptyset_1 B)(Z_t - \mu) = (1 - \theta_1 B) a_t,\ a_t \sim N(0,\ \sigma_a^2)$$

$$Z_t - \mu = \frac{1-\theta_1 B)}{1- \emptyset_1 B)} a_t = \psi(B)\, a_t$$

Where; ψ(B) is the polynomial function of order (1) in (B).

We can find the relationship between the parameters (θ) and (□) as follows:

$$\psi_j = \phi_1 \psi_{j-1} = \phi_1^{j-1}(\phi_1 - \theta_1), j \geq 1, |\phi_1| < 1 \; and \; \phi_1 \neq \theta_1$$

$$\therefore \mathbf{Z_t} = \mathbf{a_t} + (\boldsymbol{\phi_1} - \boldsymbol{\theta_1}) \sum_{j=1}^{\infty} \boldsymbol{\phi_1^{j-1}} \mathbf{a_{t-j}} \quad \dots (4)$$

4 New Identification method of ARMA (1, 1) process:

We know there are many methods using for identification the rank of mixed ARMA (1, 1) process, the best of these methods depend on extended sample autocorrelation (ESACF) which suggested by (Tsay and Tiao – 1984), this method depend on the in depended estimation (OLS) method of autoregressive with order (p) model as follows

$$\mathbf{Z_t} = \sum_{L=1}^{p} \phi_{L(P)}^{(0)} \mathbf{Z_{t-L}} + \boldsymbol{\varepsilon}_{p,t}^{(0)} \quad \dots (5)$$

Where (Z_t) is the time series of (n) observations has ARMA (p, q), (0) is the ordinary regression; (p) is the rank of model and $\varepsilon_{p,t}^{(0)}$ is the purely random error. But the second method of identification ARMA (1, 1) model is suggested by (Kuldeep Kumar – 1987), this method depend on Smoothing approach of (pade) of estimation method of ARMA (1, 1) by using autoregressive or moving average model The new method of identification ARMA (1, 1) model is suggested to extended sample partial autocorrelation (ESPACF) by using the following formula:

$$\phi_{kk} = \begin{cases} \hat{\rho}_1 & \text{if } k = 1 \\ \dfrac{\hat{\rho}_k - \sum_{j=1}^{k-1} \hat{\phi}_{k-1,j}\, \hat{\rho}_{k,j}}{1 - \sum_{j=1}^{k-1} \hat{\phi}_{k-1,j}\, \hat{\rho}_j} & \text{if } k = 2, 3, .. \end{cases} \quad \dots (6)$$

And $\hat{\phi}_{k,j} = \hat{\phi}_{k-1,j} - \hat{\phi}_{kk} * \hat{\phi}_{k-1,k-j}$ for j = 1, 2, …, k-1

We can definite the extended sample partial Autocorrelation function (ESPACF) of time series (Z_t) in general and for any positive number (m) as follows:

$$\hat{\phi}_j(m) = \hat{\phi}_j(w_{m,t)}(j)) \quad \dots (7)$$

Where $w_{(m,t)}(j) = Z_t - \sum_{L=1}^{m} \hat{\phi}_{L(M)}(j)\, Z_{t-L}$

So if the original model is ARMA (p, q), therefore $\{w_{(m,t)}(j)\}$ close to AR (P), because

$$\hat{\phi}_j(m) = \begin{cases} 0 \text{ for } m \geq p, j \geq q \\ \neq 0 \text{ other wise} \end{cases} \quad \dots (8)$$

5 Distribution of time series $\{Z_t$ = ARMA (1, 1)$\}$:

Using the characteristic function to find the distribution of $\{Z_t\}$ depend on The relationship between the characteristic function of $\{Z_t\}$ and characteristic function of white noise $\{a_t\}$, where the relationship is:

$$\psi_z(s) = \psi_a(s) \prod_{j=1}^{\infty} \psi_a \{\phi^{j-1}(\phi - \theta)s\} \quad \dots (9)$$

Where $\psi_z(s)$ is the characteristic function of $\{Z_t\}$ and $\psi_a(s)$ is the characteristic function of white noise $\{a_t\}$.

First: when the white noise $\{a_t \sim N(0, \sigma_a^2)\}$:

In this case we have:

$$\psi_a(s) = \exp\{-\tfrac{1}{2}s^2\sigma_a^2\} \quad \dots (10)$$

And by substitute in (10) we gets

$$\psi_z(s) = \exp\left(-\frac{1}{2}s^2\sigma_a^2\right) \prod_{j=1}^{\infty} \exp\left[-\frac{1}{2}s^2\sigma_a^2(\phi - \theta)^2\phi^{2j-1}\right]$$

$$= \exp\left(-\tfrac{1}{2}s^2\sigma_a^2\right)\exp\left[-\tfrac{1}{2}s^2\sigma_a^2(\phi - \theta)^2 \sum_{j=1}^{\infty}\phi^{2j-1}\right]$$

$$= \exp\left[-\tfrac{1}{2}s^2\sigma_a^2\{1 + (\phi - \theta)^2\}\sum_{j=1}^{\infty}\phi^{2j-1}\right]$$

$$= \exp\left[-\tfrac{1}{2}s^2\sigma_a^2\{1 + (\phi - \theta)^2\}(1 + \phi^2 + \phi^4 + \cdots)\right]$$

$$= \exp\left[-\tfrac{1}{2}s^2\sigma_a^2\left\{1 + \frac{(\square - \theta)^2}{1 - \square^2}\right\}\right]$$

$$\therefore$$

$$\psi_z(s) = \exp\left[-\tfrac{1}{2}s^2\sigma_a^2\left\{\frac{1 - 2\phi\theta + \theta^2}{1 - \phi^2}\right\}\right] \quad \dots (11)$$

Therefore (11) represent the characteristic function of $\{Z_t\}$ normal Distribution with mean (zero),

Variance = $\left[\sigma_a^2\left\{\frac{1 - 2\square\theta + \theta^2}{1 - \square^2}\right\}\right]$ and (pdf) as follows:

$$f(Z) = \begin{cases} \dfrac{1}{\sqrt{2\pi\left(\frac{1 - 2\phi\theta + \theta^2}{1 - \phi^2}\right)\sigma_a^2}} \exp\left[\dfrac{-Z^2(1 - \phi^2)}{2(1 - 2\phi\theta + \theta^2)\sigma_a^2}\right], & -\infty < z < \infty \\ 0 & \text{ow} \end{cases} \quad \dots\dots (12)$$

Second: when the white noise $\{a_t \sim$ poisson with $(\lambda)\}$:

In this case we have the (pmf) as follows:

$$f(a_t) = \begin{cases} \lambda^{a_t}\exp(-\lambda)/a_t! \\ 0 \quad \text{o.w.} \end{cases}, \; a_t = 0, 1, 2, \ldots \text{ and } \lambda > 0$$

So that the characteristic function of white noise $\{a_t\}$ is:

$$\psi_a(s) = \exp\{\lambda(\exp(i\,s) - 1)\}$$

Or $\psi_z(s) = \psi_a(s) \prod_{j=0}^{\infty} \psi_a(s)$

Therefore; the characteristic function of $\{Z_t = \text{ARMA}(1,1)\}$ with white noise $\{a_t \sim \text{poi}(\lambda)\}$ is:

$$\boldsymbol{\psi_z(s) = \exp\lambda\left\{\left(\exp(i\,s) - 1\right) + \sum_{j=1}^{\infty}\left(\exp\left(i\,s\,\phi^{j-1}\,(\phi - \theta)\right) - 1\right\}\right.} \quad \dots (13)$$

6 The moments of time series $\{Z_t$ when $a_t \sim poi\ (\lambda\}$:

According to above formula (13) and by using the characteristic function of $\{Z_t\}$ we have:

$$\acute{\psi}(s)/s = 0 = \exp(0)\{(i\lambda) \sum_{j=0}^{\infty} \psi_j \exp(0)\}$$

$$\therefore \quad \acute{\psi}_z(s)/s = 0 = (i\lambda) \sum_{j=0}^{\infty} \psi_j \quad \dots (14)$$

By the same way we derive the characteristic function of (second, third, and forth. To find another moment of time series (Z_t) when white noise $\{a_t \sim poi\ (\lambda\}$, so, the mean and variance of (Z_t) is: $\mu_{Z_t} = E(Z_t) = \lambda \sum_{j=0}^{\infty} \psi_j$ and $Var(Z_t) = \lambda \sum_{j=0}^{\infty} \psi_j^2$ By substitute in weights (ψ_j) of {ARMA (1, 1)}, we gets four moments of time series (Z_t) as follows:

$$E(Z_t) = \lambda\{1 + (\emptyset - \theta) \sum_{j=1}^{\infty} \emptyset^{j-1}\} \quad \dots (15)$$

$$\mathbf{E(Z_t^2)} = \boldsymbol{\lambda}\{\mathbf{1} + (\boldsymbol{\emptyset} - \boldsymbol{\theta})^2 \sum_{j=0}^{\infty} \emptyset^{2(j-1)}\} + \boldsymbol{\lambda}^2\{\mathbf{1} + (\boldsymbol{\emptyset} - \boldsymbol{\theta}) \sum_{j=0}^{\infty} \emptyset^{j-1}\}^2 \quad \dots (16)$$

$$\mathbf{E(Z_t^3)} = \boldsymbol{\lambda}\{\mathbf{1} + (\boldsymbol{\emptyset} - \boldsymbol{\theta})^2 \sum_{j=1}^{\infty} \emptyset^{3(j-1)}\} + 3\boldsymbol{\lambda}^2\{\mathbf{1} + (\boldsymbol{\emptyset} - \boldsymbol{\theta}) \sum_{j=1}^{\infty} \emptyset^{j-1}[\mathbf{1} + (\boldsymbol{\emptyset} - \boldsymbol{\theta})^2 \sum_{j=0}^{\infty} \emptyset^{2(j-1)}]\} + \boldsymbol{\lambda}^3\{\mathbf{1} + (\boldsymbol{\emptyset} - \boldsymbol{\theta}) \sum_{j=0}^{\infty} \emptyset^{j-1}\}^3 \quad \dots (17)$$

7-Skewness and Kurtosis coefficients of (Z_t) when $\{a_t \sim N(0, \sigma_a^2)\}$:

The skewness (Sk) is a measure of symmetry and all symmetric distributions have zero skewness, so

$$\textbf{Sk.} = \frac{\mu_3}{(\sqrt{\mu_2})^3} = \frac{E(Z_t-\mu)^3}{\{E(Z_t-\mu)^2\}^{3/2}} = \frac{E(Z_t^3) - 3E(Z_t^2)E(Z_t) + 2(E(Z_t))^3}{\{E(Z_t^2) - (E(Z_t))^2\}^{3/2}} \quad \dots (18)$$

But the kurtosis (Ku) is a measure peakedness of the (pdf) or the(pmf) (Usually, compared to the normal distribution), for the normal distribution, this value equal to zero; so

$$\textbf{Ku} = \frac{\mu_4}{(\mu_2)^2} = \frac{E(Z_t-\mu)^4}{\{E(Z_t-\mu)^2\}^2} = \frac{E(Z_t^4) - 4E(Z_t^3)E(Z_t) + 6E(Z_t^2)(E(Z_t))^2 - 3(E(Z_t))^4}{\{E(Z_t^2) - (E(Z_t))^2\}^2} \dots (19)$$

By using the previous four moments of time series $\{Z_t\}$ with formula (18) and (19) we find the skewness and kurtosis of $\{Z_t\}$as follows:

$$Sk:(Z_t) = 0 \quad \dots (20)$$

$$Ku:(Z_t) = 3 \quad \dots (21)$$

8- Skewness and Kurtosis coefficients of (Z_t) with $\{a_t \sim poi\ (\lambda)\}$

By the same way we can find the skewness and kurtosis of $\{Z_t\}$ as follows:

$$Sk.(\mathbf{Z_t}) = \frac{\{1 + (\emptyset - \theta)^3 \sum_{j=1}^{\infty} \emptyset^{3(j-1)}\}}{\sqrt{\lambda}\{1 + (\emptyset - \theta)^2 \sum_{j=1}^{\infty} \emptyset^{2(j-1)}\}^{3/2}} \quad \dots (22)$$

And

$$Ku.(\mathbf{Z_t}) = 3 + \left\{\frac{1 + (\emptyset - \theta)^4 \sum_{j=1}^{\infty} \emptyset^{4(j-1)}}{\lambda\{1 + (\emptyset - \theta)^2 \sum_{j=1}^{\infty} \emptyset^{2(j-1)}\}^2}\right\} \quad \dots (23$$

9- Experimental Part:

In this part we use many experimental with simulation. We expressed by using basic form as given by (Box and Muller). We find the random variable (a_t), values (Z_t) and with sample sizes (n = 50, 100 and 200) we get the follows:

A- Comparing between skewness theoretical normal and Poisson (T.N), (T.P) with experimental of normal and Poisson (E.N), (E.P) and kurtosis coefficients of (Z_t) with normal and Poisson also.

When $\{a_t \sim N(0,1)$ and $poi\ (\lambda)\}$,

We find the moments of experimental time series (Z_t) and according to formulas (18) and (19) with normal and by using (22) and (23) with Poisson we gets the following tables:

Table (1-A): The skewness coefficients of ARMA (1, 1) when { $a_t \sim N(0,1)$ and $poi\ (\lambda)$}

n	$(\emptyset, \theta)$	Sk. (Z_t)			
		T. N.	**T. P.**	**E.N.**	**E.P.**
50	**(-0.8, -0.9)**	**0**	**0.96**	**0.009**	**0.94**
100		**0**	**0.96**	**0.003**	**0.96**
200		**0**	**0.96**	**0.002**	**0.96**
50	**(-0.3, 0.1)**	**0**	**0.74**	**0.004**	**0.72**
100		**0**	**0.74**	**0.002**	**0.73**
200		**0**	**0.74**	**0.002**	**0.73**
50	**(0.3, -0.1)**	**0**	**0.84**	**0.002**	**0.82**
100		**0**	**0.84**	**0.002**	**0.83**
200		**0**	**0.84**	**0.002**	**0.84**
50	**(0.7, 0.5)**	**0**	**0.91**	**0.002**	**0.90**
100		**0**	**0.91**	**0.001**	**0.88**
200		**0**	**0.91**	**0.001**	**0.90**
50	**(0.9, 0.9)**	**0**	**1**	**0.003**	**0.99**
100		**0**	**1**	**0.001**	**0.99**
200		0	1	0.001	1.00

Table (1-B): The kurtosis coefficients of ARMA (1, 1) when { $a_t \sim N(0,1)$ and poi (λ)}

n	(∅, θ)	K.U.(Z_t)			
		T.N	**T.P.**	**E.N.**	**E.P.**
50	**(-0.8, -0.9)**	**3**	**3.95**	**3.04**	**3.89**
100		**3**	**3.95**	**3.04**	**3.94**
200		**3**	**3.95**	**3.03**	**3.95**
50	**(-0.3, 0.1)**	**3**	**3.74**	**3.06**	**3.77**
100		**3**	**3.74**	**3.05**	**3.75**
200		**3**	**3.74**	**3.00**	**2.75**
50	**(0.3, -0.1)**	**3**	**3.74**	**3.04**	**3.73**
100		**3**	**3.74**	**3.00**	**3.74**
200		**3**	**3.74**	**3.01**	**3.77**
50	**(0.7, 0.5)**	**3**	**3.86**	**3.00**	**3.81**
100		**3**	**3.86**	**3.00**	**3.80**
200		**3**	**3.86**	**3.01**	**3.86**
50	**(0.9, 0.9)**	**3**	**4**	**3.00**	**3.99**
100		**3**	**4**	**3.00**	**3.99**
200		3	4	3.01	4.55

But when ($\lambda = 20$) and (n =200) we get the following tables:

Table (2-A): The skewness coefficients of ARMA (1, 1) when{$a_t \sim$ poi ($\lambda = 20$)}

(∅, θ)	Sk. (Z_t)		Sk.(a_t)
	Theoretical	experimental	Theoretical
(0.9, 0.9)	0.22361	0.22816	0.22361
(-0.3, 0.1)	0.15868	0.15920	0.22361

Table (2-B): The kurtosis coefficients of ARMA (1, 1) when{$a_t \sim$ poi ($\lambda = 20$)}

(∅, θ)	Ku. (Z_t)		Ku.(a_t)
	Theoretical	experimental	Theoretical
(0.9, 0.9)	3.05	3.06738	3.05
(-0.3, 0.1)	3.03649	3.03756	3.05

B: Comparing between three methods (ESACF, C- table and ESPACF) of ARMA (1,1) with different parameters which identifies the stationary and invertibility values of (∅) and (θ), we gets the following table:

Table (3): Frequency distribution for ARMA (1, 1) with different methods

n	(∅, θ)	ESACF	C- table	ESPACF
50	(-0.8, -0.8)	481	478	479
100		492	481	488
200		497	486	495
50	(-0.4, 0.2)	484	480	478
100		491	489	493
200		496	497	499
50	(0.3, -0.1)	482	482	481
100		443	487	492
200		498	489	497
50	(0.8, 0.6)	485	483	487
100		492	485	493
200		499	496	495
50	(0.9, 0.9)	484	481	484
100		495	486	494
200		497	489	496

So from above table, the results of method (ESPACF) and method (ESACF) gives more iterations and much better than (C-table) method.

C: Comparing between the three methods for efficiency ARMA (1, 1) model by using (percentage Error) measurement, we gets the following table:

Table (4): the percentage Error ratio for different methods

N	**(∅, θ)**	**ESACF**	**C- table**	**ESPACF**
50	(-0.8, -0.8)	0.038	0.044	0.042
100		0.016	0.018	0.024
200		0.006	0.008	0.01
50	(-0.4, 0.2)	0.032	0.04	0.044
100		0.018	0.022	0.014
200		0.008	0.006	0.002
50	(0.3, -0.1)	0.036	0.036	0.038
100		0.014	0.026	0.016
200		0.004	0.002	0.006
50	(0.8, 0.6)	0.03	0.034	0.028
100		0.016	0.03	0.014
200		0.002	0.008	0.01
50	(0.9, 0.9)	0.032	0.038	0.032
100		0.01	0.028	0.012
200		0.006	0.01	0.008

10 CONCLUSION

1- We find the results of method (ESPACF) and method (ESACF) gives more iterations and much better than (C-table) method
2- Find that Error ratio decreasing with increasing sample size and method (ESACF) is better than the methods (ESPACF) and (C- table) respectively.

REFERENCES

[1]. J. Andel, “Marginal distribution of Autoregressive Processes Ninth Prague Conference of Information theory, Statistical Decision function and Random Process," Czechoslovak Academy of Sciences, Institute of Information theory and Automation Prague- (1982).

[2]. G.S. Fish man, “Concepts and Methods in discrete event Digital Simulation “, John Wiley and sons, New York, 1973

[3]. B. Fernandez, J.D. Salas, “Periodic Gamma Autoregressive Processes for Operational Hydrology”, Water Resources Research, 22(10), 1385-1396,1986.

[4]. B. Fernandez , J. D. Salas, “Gamma Autoregressive Models for Stream- Flow Simulation”. Journal of Hydraulic Engineering, 116 (11), 1403-1414, 1990.

[5]. H.L Nelsn. & C.W. Granger, “Experience with using the Box - Cox Transformation whom forecasting Economic Time Series”, J. of Econometrics, vol. (8), No. (1), 1979.

[6]. J.T.B. Obeysekera, V. Yevjevich, “A Note on Simulation of Samples of Gamma–Autoregressive Variables”, Water Resources Research, 21(10), 1569-1572, 1985

[7]. C.H. Sim, "A mixed Gamma ARMA (1, 1) Model for River Flow Time Series ", water Resources Research, Vol. (23), No. (1), pp. (32-36), 1987.

[8]. A.L. Swift, & G.J. Janacek, “Forecasting Non – Normal Time Series, “J. of forecasting, Vol. (10), No. (3) 1991

[9]. K. Kumar, “Identification of ARMA models using Pade Approximation “, Bulletin of the International Stat. Institute (Netherlands), Vol. (L π), Book 2, p.(377), 1987.

[10]. R.S. Tsay, and G.C. Tiao, “Consistent Estimates of Autoregressive Parameters and Extended sample Autocorrelation function for Stationary and Non–Stationary ARIMA models “, JASA, Vol. (79), No. (384),p (84), 1984.