Bernoulli Equations in the Light of the Kinetic Theory of Liquids Supposed the Theoretical Derivation of the Viscosity Equation

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Abstract: The method of a certain liquid viscosity study has been developed on the base of the force balance level. At the same time, consideration of the process at the energy balance level with the theoretical derivation of the viscosity equation made it possible to clarify the mechanism of its manifestation and propose a technique for calculating pressure differences in channels of different sizes. It takes into account even abnormal manifestations of viscosity. The theoretical derivation of the viscosity equation revealed a uniform pattern for all the liquids studied. Thus it manifests itself in the fact that the same viscosity value should be observed at the same pressure or the same shear stress under the condition of the same duration of the liquid exposure to a load. It is also shown that Bernoulli's theorem on the energy of a moving liquid particle is applicable due to the fact that, as a result of mathematical transformations, characteristic of specific energy were introduced. The particle energy has been determined by means of elementary volume and elementary mass and theoretical derivation of the viscosity equation.

Index Terms: liquids, viscosity, density, energy balance, Bernoulli equation, liquid particles

1. INTRODUCTION

As it is known, the classical approach to the study of the properties exhibited by deformed liquids is based on Euler equations of motion derived for perfect liquids that are incapable of withstanding or having any tangential shear stresses or tearing forces [1].

1.1 Purpose of the study

Meanwhile, all real liquids have a certain viscosity, which predetermines the possibility of not only normal pressures, but also other stresses. And since neither the magnitude nor the direction of the resistance due to viscosity is known, Euler chose to consider the motion of a perfect liquid and compare the results with the actually observed properties. All differences should be attributed to the effect of viscosity, and in this way, it will be possible to form an idea of what determines the viscosity and other properties of a certain liquid.

1.2 Viscosity of liquids

Viscosity is determined for each liquid experimentally as the dependence of shear stress τ on relative shear rate γ for a steady flow:

$$\mu = \frac{\tau}{\gamma^{/}} \quad (1)$$

where shear stress $\tau=F/S$ determines the liquid resistance to external force F per unit of area S, at which the relative shear of $\gamma=l/H$ layers occurs, the evaluation of which over time t leads to finding the rate of the relative displacement of the liquid particles $\gamma^{/}=\gamma/t=l/H{\cdot}t$

In fact, the formulation of the internal friction definition, adopted by Newton [2], and, accordingly, the consideration of the mechanism of the liquids' resistance to external influences by both Newton and Euler were reduced to considering the process of interaction at the level of the balance of forces. This approach was used to predetermine the very method of studying the flow of liquids in capillary [3–6] and rotational [7–10] viscometers and their design features.

Since various deviations arise in the determination of liquid viscosity, in many cases they should predetermine errors in calculating the given parameters based on both continuum mechanics [11–15] and the approach developed by the kinetic theory of liquids [16, 17]. The study of viscosity as a mechanism of resistance of kinetic units (KU) of a liquid to external action, taking into account certain provisions of the kinetic theory, led to the consideration of this process at the level of the balance of the energy of KUs' internal resistance and the external energy. This made it possible to derive theoretically the viscosity equation [18] and to determine the mechanism of manifestation of its properties in a wide range of deformation situations, including the range, in which liquids demonstrate abnormal properties.

The choice of the basic provision adopted in the derivation of the viscosity equation was also influenced by Bernoulli's theorem of the energy reserve of a moving particle of a liquid, which determined Bernoulli's approach to the study of hydrodynamic processes. The results of this study made it possible to determine the state of a liquid particle at any point in the flow [19] by switching over to different assessment systems.

2. STUDY OF THE LIQUID STATE

Having examined the state of the liquid between two sections of the jet as positive work and negative work, Bernoulli actually recorded the balance of external (positive) energy expended to overcome the internal

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(negative) energy of the KU resistance to shear to a new equilibrium position:

$$\frac{v_1^{\ 2}}{2g}+\frac{p_1}{\gamma}+Z_1=\frac{v_2^{\ 2}}{2g}+\frac{p_2}{\gamma}+Z_2 \quad (2)$$

where p_1 and p_2 − are static pressure; v_1 and v_2 − are flow velocities in sections S_1 and S_2; Z_1 and Z_2 − are section heights; g − is gravity acceleration; $Y=m\cdot g/[V]$ − is the relative share of liquid.

The following question arises in this case: How could the representation of a liquid as a continuous medium include the possibility to introduce the evaluation of the discrete medium deformation? And what is the connection between Bernoulli equation and the theoretical derivation of the viscosity equation, where the main characteristics are constant magnitudes: intermolecular interaction energy [ξ] $kg\cdot m^2/s^2$; particle mass [m], kg, and its volume [V], m^3, and the deformation is evaluated by the number of relative shear of liquid particles per unit of time $N/[t]$ or displaced volume [V/t], m^3/s?

2.1 Bernouli equation

Answering these questions, we should immediately note that Bernoulli equation solves the problem of the steady motion of a perfect liquid under the action of gravity, provided that the jet sections (S_1 and S_2) remain flat. And in view of the fact that the choice of sections S_1 and S_2 is arbitrary, we can rewrite the Bernoulli equation in general as:

$$\frac{v_1^{\ 2}}{2g}+\frac{p_1}{\gamma}+Z_1=\frac{v_2^{\ 2}}{2g}+\frac{p_2}{\gamma}+Z_2=\frac{v^2}{2g}+\frac{p}{\gamma}+Z=C \quad (3)$$

This implies that the sum of the three terms for any section of the water jet is constant (C). And due to the fact that each of the members represents the total energy or work of *1 kg* of the liquid, constant C can be considered as the energy contained in each kilogram of liquid, consisting of the KUs forming it. At the same time, their total energy does not change wherever these KUs are located, although this energy is composed of the sum of three variables, the dimensions of which are determined in different systems of process evaluation that arise when the total energy is referred to the KUs' mass, force, or volume ($m, F = m\cdot g, V$). As a result, the process is considered at the dimensional level of the remaining variables ($v^2, Z = h, p$), fixed in real conditions for real liquids.

To understand how Bernoulli could take into account the viscosity of real liquids, consider Bernoulli's evaluation of the deformation.

Proceeding from the fact that if a curve is drawn through a series of liquid flow points in such a way that the velocity vector of a liquid particle at each point is tangent to this curve, such a flow line characterizes the direction of a series of successively located liquid particles at a given time point. Bernoulli evaluated the deformation using the idea of a flow filament.

The latter has the following properties:

- ✓ the shape of a flow filament under steady-state motion remains unchanged over time, since in this case the streamlines limiting the surface do not change their shape and the cross-section area remains constant *S=Const;*
- ✓ the external streamlines do not enter the flow filament and streamlines contained in it do not exit from it, since the lateral surface of the flow filament is formed by streamlines, which the velocities are tangential to. As a result, the motion of particles passing through a constant section of the flow filament can be represented as a pure shear of KUs along the surface of the filament.
- ✓ the velocities at all points of the cross section of the flow filament are the same. Therefore, the flow rate of the filament is found by multiplying its cross-sectional area S by the velocity in its section $v=L/[t]$, as:

$$Q_T=[S]\cdot v=\frac{[S]\cdot L}{[t]}=\frac{V}{[t]} \quad (4)$$

Accordingly, for a constant cross-sectional area of filament [S], the flow rate through the filament is determined by its length L formed by a series of successively located KUs creating certain volume V displaced by external energy during [t] = 1 s (Fig.1).

$$V=Q_T\cdot[t]=[S]\cdot v\cdot[t]=[S]\cdot L \quad (5)$$

2.2 Formation of the flow filaments

It is the accepted idea that the structure of the liquid and the flow filament are formed by KUs that allows us to proceed from the fact that the geometry of the filament is formed by a certain number of KUs, the section radius of which is equal to r, and length L is determined by the same number of flow filaments with length l. In this case, KUs conceptually cut out by their motion volume $V=S\cdot L$ in a continuous medium (Fig. 1). All this leads to the formation of the liquid flow by flow filaments shifting relatively to each other. In turn, this leads to an evaluation of the liquid flow based on the concept of pure shear used in the derivation of the viscosity equation [18].

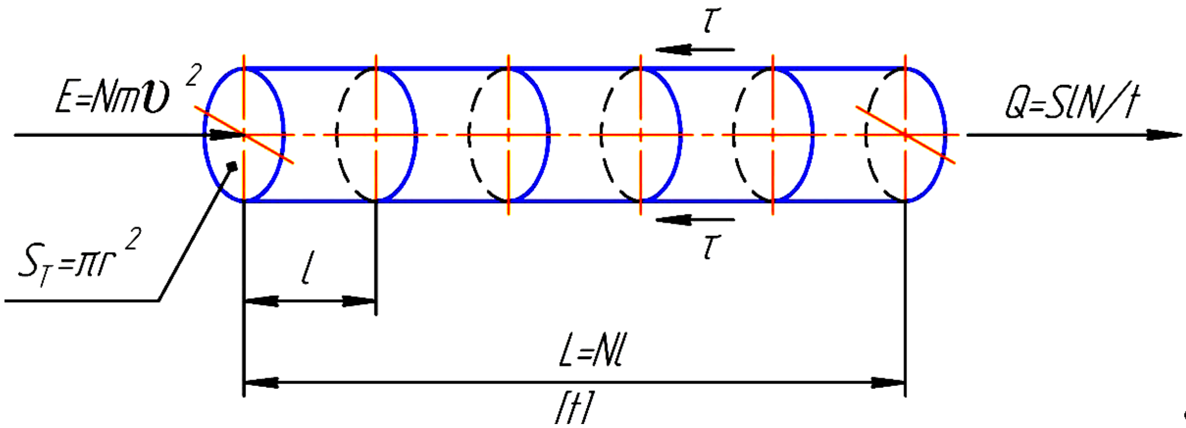

Fig. 1. *The diagram, determining the flow rate of a filament through the number of flow filaments per unit of time*

2.3 Energy of the flow

The position adopted based on the theoretical derivation of the viscosity equation [18], which follows from equation (3) and which takes into account that the energy of intermolecular interaction is constant *ξ=const*, allows determining the energy or work needed for plastic deformation of the volume isolated in the flow of volume *V* under the impact of external force *F* for a pure shear (Fig. 2) by *l*, as follows:

$$A = F \cdot l = \frac{V}{\lambda^3} \cdot \frac{l}{H} \cdot \xi \qquad (6)$$

Where $V = B \cdot L \cdot H = S \cdot H$ − is the volume of deformable liquid; $l = \lambda$ is the linear dimension of the KU of an isotropic liquid.

Based on it and provided that the deformation process lasts [*t*] = 1 *s,* we put down the following equation:

$$\frac{F \cdot l}{[t]} = N \cdot \xi \cdot \frac{l}{H \cdot [t]} = N \cdot \xi \cdot \gamma' \qquad (7)$$

where $l/H \cdot [t] = \gamma'$ − is relative shear rate; $N = V/\lambda^2$ − is the total number of KUs contained in isolated volume *V*.

Further, by referring the power balance (7) to contact surface *S* and simultaneously transforming it to the next form, we find:

$$\tau = \frac{F}{S} = \gamma' \cdot \frac{N \cdot \xi}{S \cdot l} \cdot [t] \qquad (8)$$

where $\tau = F/S$ − the tangential shear stress occurs on the contact surface $S = L \cdot B$ of the liquid layer border, when KUs resist to shear (Fig. 2).

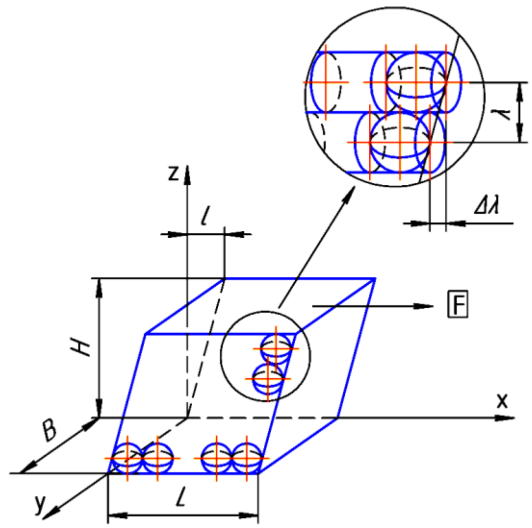

Fig. 2. *The diagram of pure shear deformation evolution*

Whence, taking into account that $N \cdot \xi = A$ − is the external work expended to deform volume $V = S \cdot H$, which creates pressure $p = A/V$, we can write down the shear stress as the following equation:

$$\tau = \gamma' \cdot \frac{A}{V} \cdot [t] = \gamma' \cdot p \cdot [t] = \gamma' \cdot \mu \qquad (9)$$

according to which viscosity is defined as the internal resistance of the KU energy to the action of external force *F*, performing the deformation characterized by relative shear rate γ'.

2.4 Connection between viscosity and pressure

This allows expressing viscosity through pressure *p* and shear stress *τ* as:

$$\mu = p \cdot [t] = \frac{\tau}{\gamma'} = \tau \cdot [t] \qquad (10)$$

Thus, in the case of the liquid flow through channels of different lengths and diameters with different velocities, the viscosity for each liquid flow rate *Q* is defined as:

$$\mu = p \cdot [t] = \frac{A \cdot [t]}{V} = \frac{A}{Q} \quad (11)$$

Knowing Q, the liquid flow rate through a certain channel with specified cross-section S and length l, and pressure p at the entering point, we can determine the energy or work as:

$$A = \mu \cdot Q = \frac{p \cdot t \cdot V}{t} = p \cdot V \quad (12)$$

In this case, the flow rate is calculated as $Q = V/t$ according to volume V, flown out within 1 second through the channel. Substituting values A and Q into equation (11), we define viscosity for a given liquid.

Solving the inverse problem for known values of viscosity μ and dimensions S and l of the pipe, through which certain water volume V is supplied for certain period, it is easy to calculate the required energy based on the first two terms of equation (12) ($A = \mu \cdot Q$). Then, expressing viscosity through pressure ($\mu = p \cdot t$) (11), we find the required pressure at the channel entrance, proceeding from:

$$A = p \cdot t \cdot Q = \frac{p \cdot t \cdot V}{t} = p \cdot V \quad (13)$$

In addition, viscosity equation (10) justifies the experimentally established [20] regularity that the same viscosity μ should be observed with the same loss of pressure p or the same shear stress $\tau(t)$ when the condition of the same duration of the material under load $l/t = v$ is met. This regularity was successfully confirmed and used in the development of a pressure difference calculation technique for polymer system flow through channels of pouring gate systems [20]. In other words, when studying viscosity at the level of energy balance, we found a single regularity inherent in all the liquids studied. At the same time, the existing method of determining viscosity at the level of the balance of forces cannot identify this regularity.
In order to discover the cause of this regularity, we express shear stress $\tau = F/S = m \cdot g/S$ through acting force $F = m \cdot g$ divided by the corresponding unit of contact surface [S], and consider the process of liquid flow at the energy balance level:

$$N \cdot \xi = F \cdot \frac{V}{[S]} = F \cdot \frac{[S] \cdot l}{[S]} = F \cdot l = F \cdot h \quad (14)$$

Where l, the displacement of liquid particles along axis x, and h, the displacement of liquid particles along axis z, are variables associated with the continuity of the deformed volume (Fig. 2).

Bearing in mind that, under the action of its own weight, the displacement of the liquid KUs takes place along axis z, the external energy is defined as follows:

$$E_h = F \cdot h = m \cdot g \cdot z \quad (15)$$

where for $F = m \cdot g = const$, only variable z determines the change of F.

2.5 Liquids density

And since the existing paradigm identifies specific weight $Y = -F/V$ of a liquid with the force referred to the volume, the force in this case:

$$F = [Y] \cdot V = g \cdot \rho \cdot V = g \cdot m \quad (16)$$

eliminates the idea of energy contained in the volume of KUs, and specific gravity Y and force F by their physical meaning are functions of bulk density:

$$\rho = \frac{Y}{g} = \frac{F \cdot g}{V} = \frac{m}{V} \quad (17)$$

characterising the particle KU weight in occupied volume V:

$$[m] = \rho \cdot V = \frac{F}{g} = \frac{m \cdot g}{g} = m = const \quad (18)$$

2.6 Energy balance

It becomes clear why Bernoulli equation was obtained based on the so-called law of living forces, which equates the kinetic energy of a moving system of material particles with mass m for a certain period with the sum of the work of all the forces acting on it for the same period, and calls it the living force:

$$E = \frac{m \cdot v^2}{2} = F \cdot l = F \cdot h \quad (19)$$

And only because Bernoulli referred the energy contained in a unit of volume to the unit of its mass [m], he managed to use the possibility of transition to the system of the discrete medium deformation process assessment, adopted in the theoretical derivation of the viscosity equation [18], through the following expression:

$$\frac{E}{[m]} = \frac{m \cdot v^2}{[m]} = \frac{m \cdot g \cdot z}{[m]} = (g \cdot z) \cdot N \quad (20)$$

in which the product of $N = m/[m]$ KUs and $g \cdot z = v^2$, m^2/s^2 characterizes the specific energy of the particle position in the flow cross section, in which $[m] = const$, and velocity $v = l/t$ is the variable.

Indeed, taking into account that in a homogeneous discrete liquid the mass of a KU and its volume are equal to $[m, V] = const$, equation (20) allows representing the energy of a single KU as follows:

$$\frac{E}{N} = \xi = [m] \cdot g \cdot z = [m] \cdot v^2 \quad (21)$$

where $\xi = E/N$ is the KU energy. Therefore, the energy of a certain KU in (21) determines only the square of its velocity v^2, omitting the constant mass of the KU from the process evaluation, which leads to the specific energy system introduced by Bernoulli.

Further, taking into account that the value of acceleration along axis z can be taken as constant $g = const$ for all liquid particles at the Earth's surface, the KU energy in the flow will depend only on value z that in the nature of the liquid field was manifested in the second system of process evaluation, the pressure head system.

$$\frac{\xi}{[m]\cdot g} = \frac{\xi}{F} = z \cdot m \qquad (22)$$

As seen from equation (21), because of the constant nature of [m] and g, their product or force $F = m \cdot g = const$ does not affect the change in energy. In this case, the KU energy is determined by a single variable, the height of the position of each KU in the flow field or geometric pressure head Z, and the system was obtained by referring the energy of KU to the force, that is, to the product of two constants $m \cdot g$.

Bernoulli equation in the third process evaluation system, the pressure system, is obtained by dividing the particle energy by its volume [V] = *const.*

$$\frac{\xi}{[V]} = \frac{[m]\cdot g \cdot z}{[V]} = \rho \cdot g \cdot z = p \qquad (23)$$

from which it derives that the energy of resistance to shear of the material particle of the liquid, adopted to evaluate the process:

$$\xi = [V]\cdot p = [V]\cdot \rho \cdot g \cdot z \qquad (24)$$

is balanced by the product of the external pressure applied to its volume (KU) $\xi = [V] \cdot p$. In this case, pressure $p = \rho \cdot g \cdot z$ is created by bulk density $\rho = [m]/V$ acting with acceleration g in the flow field with geometric pressure head z.

3. CONCLUSION

This study showed that the existing method of establishing the viscosity of a certain liquid immediately predetermines the consideration of the process at the force balance level. At the same time, the author's consideration of the process at the energy balance level with the theoretical derivation of the viscosity equation made it possible to clarify the mechanism of its manifestation and propose a technique for calculating pressure differences in channels of different sizes, which takes into account even abnormal manifestations of viscosity.

The theoretical derivation of the viscosity equation (10) revealed a uniform pattern for all the liquids studied, which manifests itself in the fact that the same viscosity value should be observed at the same pressure or the same shear stress under the condition of the same duration of the liquid exposure to a load. Thus, this same dependence $\tau(t)$ discovered much earlier as a result of analyzing experimental data of the flow of various liquids [20] was substantiated theoretically.

In addition, we managed to show that Bernoulli's theorem on the energy of a moving liquid particle became well-known and applicable due to the fact that, as a result of mathematical transformations, Bernoulli introduced the characteristic of Specific Energy, which unintentionally determined the ratio of particle energy [ξ] concluded in an elementary volume [V] to its elementary mass [m], which is inherent in the concept of the kinetic unit of a discrete liquid, enabling the theoretical derivation of the viscosity equation. It was the introduction of Specific Energy that led Bernoulli to the unintentional evaluation of deformation, taking into account the constant value of the KU interaction energy, which in fact determined the viscosity of Newtonian liquid (1), when τ and $\dot\gamma$ are directly proportional.

At the same time, we should pay special attention to the fact that the proposed property investigation method disclosed the physical meaning of Bernoulli's transformations associated with referring the energy of the liquid's KUs to constants ($m, F = m \cdot g, V$), which made it possible to switch to different systems of interaction with variables at the level of dimensions remaining after reductions, which can be used for differentiation and integration. Moreover, as will be shown in the light of the manifestation of the dissipative heating mechanism, this research method will be effective only with account of the discrete structure of liquids, which impacts the process of differentiation and integration of variables.

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