Bayesian Inference To Multiple Changes In The Variance Of AR (P) Time Series Model

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Abstract: The problem of a change in the mean of a sequence of random variables at an unknown time point has been addressed extensively in the literature. But, the problem of a change in the variance at an unknown time point has, however, been covered less widely. This paper analyses a sequence of autoregressive, AR(p), time series model in which the variance may have subjected to multiple changes at an unknown time points. Posterior distributions are found both for the unknown points of time at which the changes occurred and for the parameters of the model. A numerical example is discussed.

Index Terms: Time series model; Autoregressive model; Variance change; Posterior distribution.

1 INTRODUCTION

In recent times, inference problems associated with time series models with change point problems are increasingly met within the statistical analysis of many real life problems. In the study of change point the relationship between yield data and explanatory variables in growth models, dependence studies in chemical reactions, etc., it is very often noted that the relationship is of one type for a certain configuration of the values of the explanatory variables and of another type for a different configuration of the values of the explanatory variables. Such changes in the relationships are, some time sudden and some time gradual. In such circumstances, it is not possible to use the conventional theory of time series models which explicitly assumes a fixed rigid relationship throughout. Switching linear models are quite useful and provide better models for the data in such situations. Consider a manufacturing industry producing a particular consumer product. The profit margin of the company may follow a particular pattern (per capita) until a period when a new technology is introduced or the workers are given specialized training in handling the machines. From that period onwards the profit margin (per capita) may show a new pattern. This is an example of a sudden structural change. In this example, the time point when a structural change takes place is clearly predictable. But, in many real life problems, this may not be possible and we may have to make inferences only on the basis of the data collected on the variables of interest. In certain other situations, the structure of the models may begin to change either through the mean or variance of the errors at a particular period of time due to one or more reasons and the change may continue over a certain period of time at the end of which it might stabilize. Hsu(1977) examines the problem of testing whether there has been a change in the variance at an unknown time point using sampling theory, and applies to stock return data and also give a Bayesian treatment of a similar problem. Considerable work has been done in the recent past on the structural changes in regression models regarding the detection, estimation and inferences of switch points and parameters of switching linear regression models and sequences of normal, Poisson, Binomial and Gamma random variable. But very little work has been reported on switching time series models. Switching first order autoregressive process with one change is defined as

$$X_t = \alpha_1 X_{t-1} + e_t; \qquad t = 1, 2, \ldots, t_1$$

$$X_t = \beta_1 X_{t-1} + e_t; \qquad t = t_1 + 1, 2, \ldots, n$$

Where t_1 is the shift point, $t_1 = 1, 2, \ldots, n-1, \alpha_1$ and β_1 are the autoregressive parameter of before and after change respectively, e_t's are identically independently distributed normal variables with mean zero and common variance σ^2 and other details can be found in Broemeling (1985). The problem of variance changes in the AR(1) model is defined as

$$X_t = \beta X_{t-1} + e_t, \qquad t = 1, 2, \ldots, N$$

with $$Var(e_t) = \begin{cases} \sigma_1^2 ; & 1 \le t \le k \quad \text{and} \\ \sigma_2^2 ; & k < t \le N \end{cases}$$

and other details can be found in Menzefricke (1981).The organization of this paper is as follows. The Section I give a brief introduction about change point problems in time series models. Section II provides a brief review of change point problems. Section III provides the Bayesian inference to variance changes in the autoregressive time series models. Section IV provides the numerical study and Section V gives the brief summary and conclusion of the paper.

2 REVIEW OF LITERATURE

As was reported by Page (1950) and Anscombe were the first to investigate the switch point problem with reference to a sequence of random variables. They considered a sequence of binomial random variables and used the moving averages for estimating the point of change in the corresponding sequence of means of the random variables. The original Lindisfarne describes problem may be considered as a special case of the change point problem. A little later, as an application of the change point problem, Page (1954) investigated the problem of detecting the change in a parameter, θ, measuring the quality of an output from a continuous production process. Page also studied the problem of detecting the changes in the means of the observations in a sequence of random variables by using a cumulative sum of

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the observations (CUSUM) by assuming a distributional model. He assumed that under the null hypothesis all the n observations arise from a distribution F(x/θ) and under the alternative the first m observations come from F(x/θ′),θ≠θ′, where F(x/θ) is a member of a specified family of distributions indexed by θ. Remarking that the general properties of the indicated test procedure are difficult to evaluate, he considered the case where the variables are Bernoulli random variables and for this case he compared the power of his test with that of a standard single sample binomial test and, on the basis of numerical studies, concluded that the loss of power is quite small. Page’s single change problem was extended to the case of multiple changes by Silvey (1958) in which he formulated the problem as a multiple decision problem associated with the testing of the hypothesis that the distribution function of the i^{th} observation, say F_i, is the same for all I against the alternative that there are k changes in the sequence of distribution functions of the observations, viz.

$$F_1 = F_2 = \dots = F_i \neq F_{i+1} = \dots = F_{i_2} \neq F_{i_{2+1}} = \dots \; F_{i_{k+1}} = \dots = F_s \, , s$$

being the total number of observations. On this basis, he introduced an estimation procedure which insures that the probability of over estimating the number of changes does not exceed a specified level α whatever the true situation may be. The several works reviewed above belong to the classical or the sampling theory approach. The first Bayesian work in the area of switch point problem was by Chernoff and Zacks (1964) who introduced the Bayesian methodology to estimate the mean of the “Current variables” in multiple changes problem in a sequence of normal random variables by assuming the relationship between successive means to be

$$\mu_{i+1} = \mu_i + J_i Z_i \qquad i = 1, 2, \dots, n-1$$

Where J_i is an indicator variable which assumes the value one if the change occurs between the time points i and i+1 and zero otherwise and Z_i is a random variable which represents the amount of change when a change takes place. They assumed a normal prior for the mean and the amount of change and a uniform prior for the change points. By assuming a quadratic loss function, the Bayesian estimator for the mean of X_n was derived. He also considered the Bayesian approach to derive a test procedure for testing the hypothesis that all the μ_i ’s are equal against the alternative that $\mu_1 = \mu_2 = \dots = \mu_m \neq \mu_{m+1} = \mu_{m+2} = \dots = \mu_n$

$1 \leq m \leq n-1, \mu_{m+1} > \mu_m , m$ unknown. He considered two cases viz., μ_1 known and μ_1 unknown. He applied his test procedure to Page's problem for a binomially distributed sequence of random variables and compared the power function of the resulting test with that of the Page's test procedure numerically. The quality of the procedure was examined by a simulation study employing numerical integration technique throughout. Instead of the multiple changes model of Chernoff and Zacks (1964), Hinkley (1971) considered a model incorporating a single change in the mean of a sequence of normal random variables with constant known variance. A CUSUM test procedure was used to estimate the change point. Bayesian estimate of the change point with respect to a finite sequence of normal random variables with known σ was obtained by Broemeling (1972) who employed non informative prior for all the parameters. It is to be remarked at this stage that in the Bayesian approach to the change point problem every one, without any exception, has used only the uniform distribution as the prior distribution for the switch point. Later on, Broemeling (1974) studied the same problem employing conjugate priors for the parameters. He also used the Posterior odds ratio to test the hypothesis that the change point is a specified point. For the same model, assuming σ^2 to be unknown, Holbert (1982) derived the posterior distribution of the change point employing non informative prior for σ^2. He also carried out a numerical study with different values for the mean parameter in which shift exists. Subsequently, Lee and Heghinian (1977) studied the same change point problem for a sequence of normal random variables employing normal prior for the mean parameter and non-informative prior for the scale parameter and derived the posterior distribution of the shift point and the amount of shift. They have applied their results to data relating to traffic deaths in the state of Illinois during 1962 to 1971. Smith (1975) considered a Bayesian approach to the problem of making inferences about the point of change in a sequence of random variables at which the underlying distribution changes. The objective was, for the first time, clearly stated as locating the change point. Inferences were based on the posterior probability of the possible change points. He also gave the detailed analysis for the binomial and normal cases and also provided a numerical illustration. It is needless to point out that all the above works relate to abrupt changes in the means of a sequence of random variables. Hsu (1977) for the first time, studied the variance change problem with respect to a sequence of normal random variables and developed a locally most powerful test (LMPT) and a chi-square CUSUM test to test the hypothesis of no change against the alternative that there is a change in the variance at some point in the finite sequence of random variables. He concluded, on the basis of power comparisons, that the CUSUM chi-square test is a useful one for investigating the variance shift problem. Quandt and Ramsey (1978) introduced a kind of mixture model approach to study the changes in the means of a sequence of random variables $\{X_i\}$. They assumed that the distribution of X_i is $N(\mu_1, \sigma_1^2)$ with probability p and $N(\mu_2, \sigma_2^2)$ with probability(1-p)where $0 \leq p \leq 1$ and i = 1,2, ..., n. They also introduced a new method of estimation which consists in taking as the estimator that value which minimizes the sum of squares of the differences between the theoretical and the sample moment generating functions. He established the consistency and asymptotic normality of his estimators. Menzefricke (1981) extended the work of Hsu (1977) by considering a sequence of independent normal random variable where in the variance is subjected to a change at an unknown point of time. There were many cases corresponding to the means being equal or unequal, known or unknown, and these were discussed employing normal-gamma priors. They also gave numerical illustration to explain the evaluation of these posterior probabilities. In all the works reviewed above, one of the common basic assumptions is that the means of the random variables, before and after the switch point, were not dependent on each other. Smith (1975) examined the stability of the regression relationship over time with reference to multiple 1inear regression models. Employing non-informative prior for the change point and normal prior for the regression parameters he used the posterior odds ratio, which he called as a Bayes factor, to test the hypothesis of no change in the regression relationship against the alternative of at most one change in the regression relationship. This result was applied to certain real life data. He also explained how this technique could be applied to cover more general linear time series models and in particular, explained how a shift in the ARMA(1,1) processes can be identified by the procedure. Salazar (1980) studied the gradual changes problem in the context of time series models, first and second order autoregressive process models, lagged variable model and auto correlated error models through the Bayesian approach employing non-informative prior for all the parameters except the precision parameter for which a gamma prior was employed. Bayesian estimators were obtained for all the parameters by utilizing numerical integration techniques to remove the nuisance parameter. Venkatesan and Arumugam (2005, 2007) derived the Bayesian estimates of the first order autoregressive model through the parameter changes and Venkatesan *et al*., (2009) were studied the mean changes with respect to the autoregressive time series models with multiple changes and illustrated with numerical study subsequently Kohn and Kohn (2008) studied the multiple change points problems in mixture models.

3 MULTIPLE CHANGES IN THE VARIANCE OF AR(P) MODEL AND IT'S BAYESIAN ANALYSIS

This section is concerned with a study of the problem of changes in the variance of the AR(p) time series model and the investigation of a Bayesian inference to the same. This study was originally motivated by a study of daily stock price data, Hsu (1977) in which the time series consisted of the logarithms of the price relatives. It was apparent that, although the resulting observations were stationary in the mean, the variability was not constant throughout the series and subject to step changes at non-predictable time points. Consequently, any analysis based on the assumption of variance homogeneity could be misleading. Let us consider the general order autoregressive model, AR(p) and is given by

$$X_t = \beta_1 X_{t-1} + \beta_2 X_{t-2} + \dots + \beta_p X_{t-p} + e_t ;$$
$$t = \dots, -1, 0, +1, \dots \qquad (3.1)$$

Where $E\{X_t\} = 0 \ \forall \ t$ and all the roots of the equation $\beta(B) = \left(1 - \beta_1 B^1 - \beta_2 B^2 - \dots - \beta_n B^n\right)$ lie outside the unit circle, the Backshift operator is defined by $B^n X_t = X_{t-n}$, for which the AR(p) is stationary and $\{e_t\}$ is a sequence of independent normal variables with mean all zero and the variance σ^2 be subject to changes.

Now, consider a series $\{X_t\};\ t = 1, 2, \dots, N$ and $\hat{n}_1, \hat{n}_2, \dots, \hat{n}_m$ be preliminary estimates of the time points $n_1, n_2, \dots, n_m$ at which variance changes occur.

Define $n_0 = \hat{n}_0 = 1$ and $\hat{n}_{m+1} = N$ and

Let $\operatorname{var}(e_t) = \sigma_i^2 ;\ n_{i-1} < t \le n_i ;\ i = 1, 2, \dots, (m+1)$ and $\sigma^2 = \left(\sigma_1^2, \sigma_2^2, \dots, \sigma_{m+1}^2\right)$ given m (number of changes in the data) and $X = (X_1, X_2, \dots, X_N)'$

The likelihood function resulting from "N" observation is given by

$$P(X \mid \Theta) \propto \prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-(n_i - n_{i-1})/2} \exp\left\{-S_i(\beta)/2\sigma_i^2\right\} \dots \qquad (3.2)$$

$$S_i(\beta) = \sum_{t=n_{i-1}+1}^{n_i} \left(X_t - \beta_1 X_{t-1} - \beta_2 X_{t-2} - \dots - \beta_p X_{t-p}\right)^2 ; \quad i = 1, 2, \dots, m+1$$

$$\Theta = \left(\beta_1, \beta_2, \dots, \beta_p, \sigma_1^2, \sigma_2^2, \dots, \sigma_{m+1}^2, n_1, n_2, \dots, n_m\right)$$

Hence there are p + m + 1 + m = 2m + p +1 parameters in the model.

To find the posterior distribution of Θ, first we have to specify the prior distribution for the parameters as follows,

(i) n_i, the change points follows a discrete uniform distribution in its range.

(ii) The autoregressive parameters $\beta = \left(\beta_1, \dots, \beta_p\right)'$ follows a Multivariate normal distribution with mean vector μ and variance covariance matrix Σ (μ and Σ are known)

(iii) σ_i^2 follower inverted gamma distribution with parameters δ_i and γ_i (i = 1, 2, ..., m+1)

Therefore, the joint prior distribution of the parameters of Θ is given by

$$P(\Theta) \propto \prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-(\delta_i+1)} e^{\left(-\gamma_i / \sigma_i^2\right)} . \left|\Sigma\right|^{-\frac{1}{2}} e^{\left[-\frac{1}{2}(\beta-\mu)' \Sigma^{-1} (\beta-\mu)\right]} \qquad (3.3)$$

By using the Bayes theorem, the joint posterior distribution function can be obtained as,

$P(\Theta \mid X) \propto P(X \mid \Theta).P(\Theta)$ and is given by

$$P(\Theta \mid X) \propto \left[\prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-(n_i - n_{i-1})/2} e^{-S_i(\beta)/2\sigma_i^2}\right] \left[\prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-(\delta_i+1)} e^{-\gamma_i/\sigma_i^2}\right] e^{-\frac{1}{2}(\beta-\mu)' \Sigma^{-1} (\beta-\mu)} \qquad (3.4)$$

$$P(\Theta \mid X) \propto \prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-\frac{1}{2}(n_i - n_{i-1} + 2\delta_i + 2)} e^{-\frac{1}{\sigma_i^2}\left(\frac{S_i(\beta)}{2} + \gamma_i\right)} e^{-\frac{1}{2}(\beta-\mu)' \Sigma^{-1} (\beta-\mu)}$$

$$P(\Theta \mid X) \propto \prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-\frac{(n_i - n_{i-1})}{2} + (\delta_i+1)} e^{-\sum_{1}^{m+1}\left(\frac{S_i(\beta)}{2\sigma^2} + \gamma_i\right)} e^{-\frac{1}{2}(\beta-\mu)' \Sigma^{-1} (\beta-\mu)}$$

Where $S_i(\beta) = \sum_{t=n_{i-1}+1}^{n_i} \left(\beta_o X_t - \beta_1 X_{t-1} - \beta_2 X_{t-2} - \ldots - \beta_p X_{t-p}\right)^2$

$$= \sum_{t=n_{i-1}+1}^{n_i} \left(\sum_{0}^{p} \beta_r^2 X_{t-r}^2 + 2 \sum_{\substack{r,s=1 \\ r<s}}^{p} \beta_r \beta_s X_{t-r} X_{t-s} \right)$$

$$= \sum_{t=n_{i-1}+1}^{n_i} \sum_{r=0}^{p} \beta_r^2 X_{t-r}^2 + 2 \sum_{t} \sum_{\substack{r,s=1 \\ r<s}}^{p} \beta_r \beta_s X_{t-r} X_{t-s}$$

and $\frac{S_i(\beta)}{2} + \gamma_i = \frac{1}{2}\left[S_i(\beta) + 2\gamma_i\right]$

$$= \frac{1}{2}\left[\sum_{t=n_{i-1}+1}^{n_i} \left(\beta_0 X_t - \beta_1 X_{t-1} - \ldots - \beta_p X_{t-p}\right)^2 + 2\gamma_i \right]$$

$$= \frac{1}{2}\left[\sum_{t} \sum_{r=0}^{p} \beta_r^2 X_{t-1}^2 + 2 \sum_{t} \sum_{\substack{r,s=0 \\ r<s}}^{p} \beta_r \beta_s X_{t-r} X_{t-s} + 2\gamma_i \right]$$

$t = n_{i-1} + 1, \ldots, n_i; \quad r = 0, 1, \ldots, \mathrm{P}; \quad \mathrm{i} = 1, 2, \ldots, (\mathrm{m}+1)$

When, we define $A_i^{'} = \Sigma$, then we get the following, after simplification,

$$P(\Theta \mid X) \propto \left[\prod_{i=1}^{m+1} \left(\sigma_i^2\right)^{-\frac{(n_i - n_{i-1})}{2} + (\delta_i + 1)} \right] e^{-\frac{1}{2}(\beta-\mu)^{'} \Sigma^{-1} (\beta - \mu)}$$

$$\exp\left[-\sum_{i-1}^{m+1} \left\{ \sum_{t=n_{i-1}+1}^{n_i} \sum_{r=0}^{p} \beta_r X_{t-r}^2 + 2 \sum_{t} \sum_{r,s=0}^{p} \beta_r \beta_s X_{t-r} X_{t-s} + 2\gamma_i \right\} \right] \quad (3.5)$$

The above joint posterior distribution of

$\Theta = \left(\beta_1, \beta_2, \ldots, \beta_p, n_1, n_2, \ldots, \mathrm{n}_m, \sigma_1^2, \ldots, \sigma_{m+1}^2\right)$ is a very complicated expression and is analytically not tractable. One way of solving this problem is to find the marginal posterior distribution using MCMC technique.

4 NUMERICAL STUDY

In order to illustrate the solutions of the change point problems described in section 3 a computer study was carried out. The main aim of the numerical study is to illustrate the evaluation of the estimates of the parameters on the basis of the methodology developed. In this paper MCMC technique is used to compute marginal posterior distributions and Bayes estimates of the parameters. Markov Chain Monte Carlo (MCMC) is a powerful technique for performing integration by simulation. In recent years MCMC has revolutionized the application of Bayesian statistics. Many high dimensional complex models, which were formally intractable, can now be handled routinely. Bayesian calculations not analytically tractable can be performed using MCMC once a likelihood and prior are given. MCMC simulation algorithm in its basic form is quite simple and becoming standards in much Bayesian application. One of the basic goals of general Bayesian framework is to compute expectations with respect to a high dimensional probability distribution. The two main algorithms used in MCMC applications are (i) Gibbs Sampler and (ii) Metropolis algorithms. All other algorithms are generalization of Metropolis. The simplest MCMC algorithm is a Gibbs sampler. Gibbs sampler iteratively samples each variable conditional on the most recent value of all other variables. The analytical solution is not available for the posterior density function since the joint posterior distribution is a complicated function of the parameters. Therefore, one may have to resort to numerical integration or MCMC or Gibbs sampling technique to determine the estimates. Variance changes in time series data is illustrated by following. Box and Jenkins (1970) fitted the ARIMA (0, 1, 1) model to a series of 369 IBM stock prices. The same data is fitted through the methodology suggested in section 3 by the first difference of the same series (logarithm). Further due to certain practical limitations in computing, attention was focused on the switching first order autoregressive process. The point estimates of the parameters were evaluated numerically by taking the posterior mean as the estimate. MCMC technique is used to evaluate the marginal posterior distribution and hence the Bayes estimates. We propose an AR(1) model with two innovations variance changes and is given in the following results:

$$\hat{\alpha} = 0.13, \qquad 10^4 \operatorname{var}(e_t) = \begin{cases} 0.99 & t < 179 \\ 0.60 & 180 \le t \le 235 \\ 0.72 & t \ge 235 \end{cases}$$

5 RESULTS AND CONCLUSION

This paper is the output of an investigation regarding the building up of suitable models to represent the structural changes in time series models. Bayesian methodology has been used in making inferences about the parameters of the model. A Bayesian solution of the change point problems through the variance change in autoregressive models is discussed and a numerical study has been illustrated through the procedure developed and also to examine the quality of the results obtained. Generally, the estimates are found to be close to the true values using which IBM stock price data. The estimates are quite close to the true values when the magnitude of the switch is large relative to the variance.

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