Solving Linear And Non-Linear Partial Differential Equations Using Numerical Techniques

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Abstract: Advancement in technology and engineering poses us with many challenges, similarly in order to overcome such engineering challenges with the help of numerous mathematical models, equations are taken. Since in the beginning Mathematicians, Designers and Engineers strive for correctness and accuracy while solving equations Differential equations, in particular, hold a tremendous application in engineering and many other domains. One such type of Differential equation is known as partial differential equation. The spectrum of application of partial differential equations consists of simulation, algorithm generation, and analysis of higher order PDE and wave equations. Adapting different numerical methods leads to a variety of answers and difference among them, therefore the selection of the method of solving is one of the crucial parameters to generate precise results. Our work focuses on the review of different numerical methods in order to solve Linear and Nonlinear differential equations on the basis of accuracy and efficiency, so as to reduce the iterations. These would be harmonizing guidelines to existing numerical methods of linear and nonlinear partial differential equations.

Key Words: Linear and Non-Linear Partial differential equations, Numerical methods, Iterations and Simulation.

1 INTRODUCTION

Partial differential equations (PDE) have found a spectrum of applications in engineering, design and mathematical modeling in the past decade. The complexity of problems say in the field of engineering and codomains are increasing day by day and we are facing issues to solve them precisely and in the limited amount of time frame. The major mathematical division in partial differential equations are Linear and Nonlinear Partial differential equations. Linear PDE can be defined in which all the terms and any of its derivatives can be expressed as a linear combination in which the coefficients of the terms are independent. Moreover, the coefficients can depend most on the independent variables. A linear PDE can be of different degree orders i.e. first-degree order, second-degree order, etc. In contrast to this, a non-linear PDE is a partial differential equation with non-linear terms. Non-linear PDE is described in many different physical systems involving of gravitation, fluid dynamics, and mathematical modeling. Many articles have focused on demonstrating the accuracy and computational efficiency of the proposed methods in simulating the transient dynamics generated by a linear advection equation in case of linear PDEs. In this paper, the linear and non-linear PDEs that we considered were solved using different numerical methods and for different orders. In particular, certain works have discussed canonical and hierarchical tensor expansions combined with alternating least squares and high-order singular value decomposition, both implicit and explicit time integration methods are discussed. Another important study of linear PDEs includes the Laplace homotopy analysis. Also, residual power series (RPS), a new analytical iterative method for solving initial value problems of linear PDEs of fractional order is studied. On the other hand, generalized finite difference methods for non-linear PDE including higher orders is discussed, also with other works studying physical systems applications of non-linear PDEs. is also discussed, fractional PDE which is the subdivision of non-linear PDEs along with solitary wave solutions are studied as well.

2 LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Apart from non-linear PDEs which are extensively used and applied in the physical systems and mathematical modeling, high dimensional PDE occurs in various domains of engineering and especially the case of linear high-dimensional PDE is one of its kind, contributing to the numerical solution methods [11] developed a new parallel algorithm to solve such kind of PDEs. The main concept of this algorithm is based on the canonical and hierarchical numerical tensor method algorithm with at least square and hierarchical singularity value decomposition, the work includes of integration method consisting of implicit and explicit cases as well. The work demonstrated high accuracy and efficiency for simulation of the transient design generation by linear advection equation in 6 dimensions plus time, along with Boltzmann - BGK equation. Numerical technique is claimed to be very fast and accurate, but application of this method is little difficult for complex domains. Number of iterations is the crucial factor in determining the calculation time requirement, many works reported aims to bring down this calculation time and increase the accuracy of results, one such study by [12] developed a new analytical iterative method with correlation to solving initial value problem in linear PDEs of fractional order in Caputo sense. The Liouville-Caputo has the deremt that it's kernel has singularity that means this kernel consists of many effects and therefore cannot precisely describe the complete effect of memory in calculation, due to this a formulated numerical method is given by [13] to determine the approximation solution of FPDEs and is demonstrated with an analysis of combination of Laplace transform and homotopy methods in Caputo-Fabrizio and Liouville-Caputo sense. Moreover, the study demonstrates that the LHAM is very efficient tool to generate solution for linear FPDEs with fractional Caputo-Fabrizio type operators, and even gives quick convergent series solution in fewer iterations. Alternative representation of higher order PDE to singular form such as different 1st order PDEs is possible, the article by [14] provides a direct and easy way to convert a system of higher order constant coefficient linear PDE into system of 1st order PDE.

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The author states that major deciding factor is the choice of the notion of multivariate polynomial, since it can vary the results considerably. The application of linear PDEs consists of wave equations and other higher domain equations which are highly used in numerous applications, one particular kind of equation that occurs is Peridynamic equation. [15] here aims to analyze the nonlocal linear wave equation and develop efficient numerical scheme to solve linear wave equation according to Peridynamic calculation in continuum mechanics. Certain specialty of the study was done after introducing a new variable and then the original 2nd order wave equation is transferred to 1st order Hamiltonian ordinary differential equation system. The method demonstrates high precision and efficiency in 3D and is more stable over a range of large time-step and long integration, additional to this the technique is proven to be highly dominating to solve linear Peridynamic equations, the method however, cannot be directly applied for non-linear system of PDEs. While solving PDEs, we sometimes come across a finite set of homogenous PDE in various dependent and independent variables, which has solutions space of finite dimension. Such particular system leads to generation of hyper exponential solutions, [16] here presents an algorithm for computing all of the hyper exponential solutions of definite system whose coefficients are also rational function and has solution defined in a certain solution space of definite system. Their extended work also presents an algorithm to even find hyper exponential rank of the given PDE. Also, the given algorithm generalizes the computation of hyper exponential linear PDE with the help of Beke-Schlesinger algorithm. Presented algorithm suggests a way to avoid generation of too many candidates for the hyper exponential solutions.

3 NOnLINEAR PARTIAL DIFFERENTIAL EQUATIONS

As numerous works reported the application and use of non-linear PDEs in multiple domains and simulations, we aim to discuss the numerical methods which are used in different kind of PDE structures and situations in order to solve them accurately and with less complexity. As non-linear PDE holds importance in tremendous physical systems and simulation modeling, therefore, the work by [1] talks about GFDM (Generalized Finite Difference Method) for solving different parabolic non-linear PDEs, also convergence with respect to semi-linear and quasilinear equations is studied in this methodology and in addition to it limits of convergence have been developed and implemented to promote efficient solutions. The examples illustrate the viability of the application of GFDM for solving parabolic non-linear PDEs in 2D. Various cases are tested involving GFDM consisting of acoustic, heat transfer, mass transfer, and combustion systems. The results of the work validate the solution of such problems with this method also considerable efficiency is developed for the study conducted. Some of the mathematical approaches define a simpler and direct methodology in order to solve the cases of PDEs, similar direct methods are reported by [2] for constructing the exact solutions to non-linear equations of mathematical physics, these solutions involve an implicit reaction containing several free functions. A study has specifically targeted non-linear reaction-diffusion equations of generalized form, which are rather dependent on several arbitrary functions. Generalized equation as per [2] is defined as follows, \( \int h(u)du = \xi(x)\omega(t) + \eta(x) \), where the functions \( h(u), \xi(x), \eta(x), \omega(t) \) are determined in the subsequent analysis for the given equation. Certain kind of problems have the solutions falling under the category of traveling wave or wave solutions, thus supporting the same work conducted by [3] reveals a new exact traveling wave solution including solitary solution for 6th order Boussinesq equation and RLW (Regular Long Wave) equations by exp-function methods. The obtained solitary wave solutions were not reported before and are new to the work done. Results for the use of this numerical method concludes that exp-function is a powerful and efficient mathematical tool for solving non-linear evolution equations which are frequently observed in engineering, mathematical and natural science domains. Additionally [3] states that the study can be used in NLEE for any further research or investigation. Similar kind of study by [10] suggests a particular numerical method which aims to obtain solitary wave solution of a large class of non-linear PDEs, including 3 parameters as \( \alpha, \beta, \gamma \). Concluding points of the study state that if the system is non-linear algebraic equation which consists of 3 equations then the above three parameters can be expressed by the parameters of non-linear PDE, and if the system has more than 3 equations then, there is also the relationship among the parameters of solved non-linear PDE. The methodology is defined as the tool to find the exact solitary traveling wave solution for the various non-linear PDEs. As the intricacy and complexity of the modern problems are growing exponentially the mathematicians in order to solve them introduce new numerical methods. One such investigation conducted by [4] defines a new numerical method named RPS, which is used to construct and predict the exact solitary pattern for solutions in non-linear time-fractional dispersive PDEs. The technique used here is based on the generation of the residual error function and applying the Taylor series formula of generalized form. This technique not only saves time but in additionally its simpler as it depends on the recursive differentiation of time-fractional dispersive equation and use of given initial constrains to calculate coefficients of multiple FPS solution. The approach of this numerical method is free of linearization, perturbation, and discretion of variables, moreover, it is not affected by the computation of round off errors. Concluding remarks of the work outperforms the conventional methods and gives rapid solutions with better accuracy. One of the applications of PDEs is observed in the 3D physical systems, particularly for case of non-linear PDE and it sometimes even involved a particular type of PDE termed as FPDE( Fractional Partial Differential Equation) the aim to solve such type of systems, [5] defines that a 3D differential transform which is applied to system of non-linear PDE or FPDE has exactly same results which are obtained by homotopy analysis method, the methodology simplifies the complex calculations and reduces solving time. For the case of 3D FPDE, this technique is also defined, moreover FPDTM (Fractional Partial Differential Three-Dimensional Method) is rather more powerful mathematical tool to solve such 3D non-linear PDEs. Many solutions of mathematical equations have been generalized and this kind of approach helps to solve a majority of the problems in the domain of PDEs [6] reported their work which aims to develop a generalized form of solution to PDEs either it is linear or non-linear for which existence can be proved for certain boundary conditions. The method so far is only been able to solve fairly special cases of PDEs having special structures, main concept behind developing this technique is
the method of integration by parts to interpret derivatives weakly from bypassing it to the test function, moreover, the approach enables us for merely measuring maps to be rigorously studied. The technique is fairly time taking to solve but as far as the special cases of PDEs are concerned, it gives precise results to those in these in generalized format. As studying numerical solutions for different cases sometimes, we end up certain kind of solutions which requires further analysis and investigation. Such typical solution is studied by [7] which is defined as Lump solutions. The study done aims to analyze a class these Lump solutions which are developed from quadratic equations to non-linear PDE. The primary success of the Hirota bilinear formulation is the class of multivariate quadratic functions. The conclusion of the work remarks that theorem (3.2) and (3.7) used by [7] in the investigation, combined prove that Hilbert’s 17th problem for quadratic equation function, but the conjecture was not true for higher-order polynomial functions. Moreover theorem (3.2) depicts that if any quadratic function (f) is positive on (R^n) for all (X, R^n), but this is not the case for higher-order multivariate polynomials. Extending the study for PDEs, space fractional-order reaction differential equation is also investigated by [8] on a finite but large spatial domain x e [0, L] x = x (x, y) and t e [0, T], as predicted and stated by [8] the main advantage of this method results to a full diagonal representation of fractional operator and its ability to give spectral convergence despite the value of fractional power index (α) also the numerical method is efficient and reliable for the system simulation consists of Fractional-reaction diffusion problems in higher dimensions. Further, the works specifically target to the interest of non-linear wave phenomena in addition to this, it also gave the evidence that pattern formation is available in fractional scenarios, which is rather almost similar to standard reaction diffusion case.

Kudryashov’s method which is one of the most used techniques proposed to solve fractional differential equations involving higher-order derivatives and non-linear terms. The work by [9] proposed a latest version of Kudryashov’s method for solving non-integrable problems in mathematical physics, one of the sample tests concluded a new exact solution for heat conduction equation and K (m, n) equation were obtained with generalized evolution moreover the results obtained from the proposed methodology have been verified with obtained by the (G’G)-expansion method. Further highlights of the study include the symmetrical hyperbolic Fibonacci function and Lucas function solutions obtained by general Expα – function along with the Kudryashov’s method. Establishing the numerical functions of certain type of Fibonacci series, sin, cosine, tangent and cotangent functions for symmetrical Fibonacci were defined by [9], this also provided an easier solution for the problem. Similarly, for symmetrical Lucas sin and cosine are defined, even general solution can be obtained by applying modified method with symmetrical Fibonacci function, also it enables us to determine the previous solution by substitution of (α = e).

Extending the work and studies in non-linear numerical methods[17] has examined some symmetric techniques for finding local solutions for 1st order non-linear PDE for Frobenius integral vector field distance via symmetry. This method developed a technique for solving 1st order quasilinear PDEs that are applied to general 1st order non-linear PDE, results of their work defines and states significant algorithm for solving 1st order PDE, where the presence of symmetry is found that are mainly mechanical in nature. Examination of the symmetric solutions of the 1st order non-linear PDEs which are of one dependent variable and two independent variables in nature. One of the difficulty which occurs is that after the generalization of local solutions of such non-linear PDE, the resultant 1st order quasilinear PDE is considerably complicated to solve further, thus in order to solve this particular issue [17] also suggests a new vessiot integration scheme for solving such 1st order non-linear PDE, which in turn generated 1st order quasilinear PDE of only two independent variables, which demands a single solvable structure of only two symmetry. The downside was that its application was only limited to 1st order non-linear PDE. This study also provides the prevention of 2 stage process and enables us to generate solutions by simply solvable structure of symmetry.

4 CONCLUSION
We presented our work to learn different numerical methods to solve both Linear and Non-Linear PDEs which increase a standard among researchers to identify the precise numerical method to reduce the iterations and also an effective method which is suitable for their research problem. In particular, certain works have discussed canonical and hierarchical tensor expansions combined with alternating least squares and high-order singular value decomposition, both implicit and explicit time integration methods.

5 REFERENCES


