Dynamics Of A Leslie-Gower Predator-Prey Model With Holling Type Iv Functional Response And Disease In Prey

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Abstract: In this study, dynamics of a Leslie-Gower predator-prey model with Holling Type-IV functional response and with disease in prey population in presence of predator harvesting is investigated. The uniform boundedness and positivity of solutions are studied. We examine the dynamics of the system at points of equilibrium. Further, at axial equilibria, disease free equilibria and predator extinct equilibria, sufficient conditions for their existence and stability are derived. By constructing appropriate Lyapunov function, we analyze the global stability of the interior equilibrium point. To support the theoretical results, numerical simulations are performed.

Keywords: Leslie-Gower Predator-Prey model, Infectious disease, harvesting Predator, Lyapunov function.

I. INTRODUCTION

In recent times predator-prey dynamics have acted as important mechanism for understanding the complicated nature of interactions among various species in the environment. The nature of interactions among the species in the predator-prey(PP) model is called functional response(FR). After Lotka [1] and Volterra [2] pioneered the predator-prey model, many mathematical biologists contributed towards introducing different FR to interpret various interactions in the field of ecology, epidemiology, finance, defence etc. [3, 4, 5, 6, 7].
Leslie [8] proposed the model

$$\frac{du}{dt} = (r_1 - b_1 u)u - p(u)v; \quad \frac{dv}{dt} = (r_2 - \frac{a_2 v}{x})v$$

where $u(t), v(t)$ mean the population of prey and predator respectively. Significance of this model is that, according to the FR $p(u)$, predator consumes prey and there is a logistic growth of prey population with growth rate r_2. Carrying capacity $\frac{r_2 u}{a_2}$ is proportionate to the size of prey population. The term $\frac{y}{x}$ computes the depletion in the predator population due to the insufficiency of its favourite food. Aziz-Alaoui and Daper [9] has proposed a modified Leslie-Gower(L-G) model in presence of Holling type-II FR. In this model they showed that in case of rarity, predator tends to shift to different prey population but due to non availability of its favourite food, its growth will be limited. Hereafter many researchers have studied this modified L-G model to represent many different types of ecological interaction among species. A modified L-G model with Holling type II FR with time delay is analized in [10] by Nindjin et-al. A Delayed three species L-G model is studied in [11]. Yafia et-al.[12] examined limit cycle and numerical simulations in delayed L-G model. Yu [13] has analysed L-G model with Beddington-DeAngelis FR.

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In [14, 15, 16, 17] authors have investigated L-G model with harvesting. In the study of population ecology, Holling Type I, II and III FR [18, 19] have been investigated extensively in describing population interaction. In [20] authors have investigated the dynamics of L-G model with Holling type-II FR. Andrews [21] has suggested Holling Type-IV FR which is of the form $\phi(u) = \frac{mu}{1+au+bu^2}$. Bifurcation analysis of L-G model with Holling type-IV FR is studied in [22]. In population dynamics epidemiology occupies a significant place. After Kermach and Mackendric propose the S-I-R model [23], there is no stopping in employment of this model in various areas of science, finance and defense. This area of epidemiology may be classified into three categories: a) PP model having disease in prey [25, 26, 27, 28].In [25], for the first time, Anderson and May have studied predator-prey model where predator interacts with infected prey. b) PP model with disease in predator [29, 30, 31]. c) PP model where both the population are infected by disease [32, 33, 34, 35].In this work, we propose a L-G predator-prey model with Holling type-IV FR in presence of diseased prey population. A Linear predator harvesting term is added.

II.MODEL FORMULATION

A. Original model

The L-G model given in [36] is

$$\frac{dx}{dt} = (r(1 - \frac{x}{k}) - \frac{my}{\frac{x^2}{i}+x+a})x$$

$$\frac{dy}{dt} = s(1 - \frac{ny}{x})y$$

with the initial condition $x_0 > 0, y_0 > 0$, $x = x(t)$ and $y = y(t)$ are the prey and predator density respectively.

System 1 is described on the set
$\Delta = ((x, y)\epsilon R^2 / x \geq 0, y \geq 0)$

B. The L-G model with Holling Type-IV functional response with disease in prey

Based on the original model, we have formulated the following model:

$$\frac{du}{dt} = r(1-\frac{u}{k})u - \frac{\lambda_1 uw}{\frac{u^2}{i}+u+a} - \beta uv$$
$$\frac{dv}{dt} = \beta uv - \frac{\lambda_2 vw}{\frac{v^2}{i}+v+a} - \alpha v \qquad (2)$$
$$\frac{dw}{dt} = \delta w(1-\frac{nw}{u+v}) - hw$$

Table 1: Meaning of symbols

r	Rate of intrinsic of prey
k	carrying capability of prey
β	Rate of transmission of disease from infected prey to susceptible prey
λ_1	Rate of predation of susceptible prey
λ_2	Rate predation of infected prey
α	Rate of Mortality of infected prey
δ	Rate of intrinsic of predator
n	number of prey needed to support one predator at the point of equilibrium
i	A direct assessment of the predator's immunity from the prey
a	The half saturation constant in the absence of any preventing effect
h	harvesting effort

To formulate the epidemiological model we have made the following assumptions:

1. The total biomass of prey consists of two classes (a) susceptible prey $u(t)$ (b) infected prey $v(t)$ and $w(t)$. denotes the predator population.
2. Speading of disease among prey population is horizontal(i.e. by contact only) not vertical. The infected preys donot recover.
3. Both susceptible and infected preys are captured by predators. Since susceptible preys are stronger than infectious preys, so later one is more vulnerable to predation and hence $\lambda_2 > \lambda_1$.
4. In absence of the predator, susceptible prey population grow logistically.
5. Death of infected prey occurs naturally as well as by predation.
6. To express the interaction between predator and both the preys a Holling Type-IV FR is used.
7. Since the prey population is reducing by infection as well as predation, in order to protect the prey population from eradication, we have introduced harvesting hw in the system 2.

III. BOUNDEDNESS AND POSITIVITY OF SOLUTIONS

The importance of examining the boundedness and positivity of the solution of a PP system is that, positivity makes sure that the population survive in future and boundedness tells us about some kind of restriction on growth due to limited resources. In case of LG model, boundedness may be dependent on the lack of favourite food.

Lemma 3.1 If $a,b > 0$ and $\frac{dP}{dt} \leq (\geq) P(a-bP)$ with $P(0) > 0$ then $\overline{\lim}_{t\to\infty} P(t) \leq \frac{a}{b}$ $(\underline{\lim}_{t\to\infty} P(t) \geq \frac{a}{b})$

Lemma 3.1 may be presented equivalently in the form of following Lemma 3.2.

Lemma 3.2 If $a,b > 0$ and $\frac{dP}{dt} \leq (\geq) P(a-bP)$ with $P(0) > 0$, then for all $t \geq 0$

$$P(t) \leq \frac{a}{b-Ce^{at}},\ C = \frac{bP(0)-a}{P(0)}.$$

In Particular, $P(t) \leq \max\{P(0), \frac{a}{b}\}$ for all $t \geq 0$.

Theorem 3.1 In the region $\Delta = \{(u,v,w)\epsilon R_+^3 : W = \frac{\psi}{\mu} + \epsilon, \epsilon > 0\}$,where $\psi = \frac{k(r+\mu^2)}{4r}$ and $\mu = \min\{\alpha, h-\delta\}, h > \delta$, all the solution of the system 2 are uniformly bounded.

Proof. From susceptible prey equation we get

$$\frac{du}{dt} \leq r(1-\frac{u}{k})u$$

By Lemma 3.2, we have

$\overline{\lim}_{t\to\infty} u(t) \leq m_1$

where $m_1 = \max\{u(0), k\}$

Let us define:

$$W(t) = u(t) + v(t) + w(t)$$

This yields

$$\frac{dW}{dt} \leq r(1-\frac{u}{k})u + \delta w(1-\frac{nw}{u+v}) - hw - \alpha v$$

Now,

$$\frac{dW}{dt} + \mu W \leq r(1-\frac{u}{k})u + \delta w(1-\frac{nw}{u+v}) - hw - \alpha v + \mu(u+v+w)$$
$$\leq (r(1-\frac{u}{k}) + \mu) u + (\mu-\alpha)v + (\mu-(h-\delta)) w$$

We know that $\max\{(r(1-\frac{u}{k})+\mu)u\}$ is $\frac{k(r+\mu^2)}{4r}$

$$\therefore \frac{dW}{dt} + \mu W \leq \frac{k(r+\mu^2)}{4r} + (\mu-\alpha)v + (\mu-(h-\delta))w$$

if $\mu = \min\{\alpha, h-\delta\}, h > \delta$, then

$$\frac{dW}{dt} + \mu W \leq \frac{k(r+\mu^2)}{4r} = \psi$$

∴ From the above inequality we have, $\frac{dW}{dt} + \mu W \leq \psi$

This implies

$$0 < W(u,v,w) < \frac{\psi}{\mu}(1-e^{-\mu t}) + W_0 e^{-\mu t}$$

As $t \to \infty$, we get $0 < W(u,v,w) < \frac{\psi}{\mu}$. Therefore all the solution of the system are uniformly bounded within the region $\Delta = \{(u,v,w)\epsilon R_+^3 : W = \frac{\psi}{\mu} + \epsilon, \epsilon > 0\}$.

Theorem 3.2 All the solutions of system 2 with initial conditions are positive for all $t \geq 0$

Proof. : Let $u(t), v(t), w(t)$ be any solutions of system 2. Let us assume that any one solution of system 2 is not positive. Then any one of the following cases appears:

Case 1: If $u(t) < 0$ for some t, then there exists some time t' such that $u(0) > 0$, $u(t') = 0$, $\frac{du(t')}{dt} < 0$, $v(t) > 0$, $w(t) > 0$, $0 \leq t < t'$

Then from system 2, $\frac{du(t')}{dt} = 0$. This is contradiction to $\frac{du(t')}{dt} < 0$.

Case 2: If $v(t) < 0$ for some t then there exists some t'' such that $v(0) > 0$, $v(t'') = 0$, $\frac{dv(t'')}{dt} < 0$, $u(t) > 0$, $w(t) > 0$, $0 \leq t < t''$

Then from system 2, $\frac{dv(t'')}{dt} = 0$. This is contradiction to $\frac{dv(t'')}{dt} < 0$.

Case 3: Similarly we can establish that $w(t)$ can not be negative.

∴ The solutions $u(t), v(t), w(t)$ of the system must be positive for all $t \geq 0$.

IV. STABILITY OF EQUILIBRIUM POINTS

The points of equilibrium of the model 2 are found by solving the system of equations

$$r(1-\frac{u}{k})u - \frac{\lambda_1 uw}{\frac{u^2}{i}+u+a} - \beta uv = 0$$

$$\beta uv - \frac{\lambda_2 vw}{\frac{v^2}{i}+v+a} - \alpha v = 0$$

$$\delta w(1-\frac{nw}{u+v}) - hw = 0$$

and we find the Jacobian matrix

$$\begin{pmatrix} -\frac{2ur}{k}+r-v\beta-\frac{i(ai-u^2)w\lambda_1}{(ai+u(i+u))^2} & -\beta u^* & -\frac{i\lambda_1 u^*}{(u^*)^2+iu^*+ai} \\ \beta v^* & -\alpha+\beta u^*-\frac{i\lambda_2\left(ai-(v^*)^2\right)w^*}{\left((v^*)^2+iv^*+ai\right)^2} & -\frac{i\lambda_2 v^*}{(v^*)^2+iv^*+ai} \\ \frac{n\delta(w^*)^2}{(u^*+v^*)^2} & \frac{n\delta(w^*)^2}{(u^*+v^*)^2} & \frac{(\delta-h)u^*+(\delta-h)v^*-2n\delta w^*}{u^*+v^*} \end{pmatrix}.$$

Theorem 4.1

The trivial equlibrium $(0,0,0)$ in unstable.

1. Axial equlibrium point $(k,0,0)$ is stable if $k < \frac{\alpha}{\beta}$ and $\delta < h$

2. Predator extinct equilibrium $(\frac{\alpha}{\beta}, \frac{r(\beta k-\alpha)}{\beta^2 k}, 0)$ exist if $k > \frac{\alpha}{\beta}$ and is stable if $\delta < h$. $k > \frac{\alpha}{\beta}$ is also a condition for stability.

3. Disease free equilibrium points exists if $\delta > h$ and are stable if $(h-\delta)(i^2k\lambda_1 u^*(2a+u^*)(h-\delta) + \delta nr(k-2u^*)(ai+u^*(i+u^*))^2) > 0$, $\delta n(ai+u^*(i+u^*))^2(-hk+k(\delta-r)+2ru^*) - ik\lambda_1 u^*(h-\delta)(ai-(u^*)^2) > 0$

and $\frac{-a\delta n(\alpha-\beta u^*)-\lambda_2 u^*(\delta-h)}{a\delta n} > 0$.

4. Internal equilibrium point (u^*, v^*, w^*): If carrying capacity k is greater than i then exists atleast one (+)ve root of u^*. Also if $2\sqrt[3]{2}\,\delta nr\left(\delta nr\left(-3ai+i^2+ik+k^2\right)+3ik\lambda_1(h-\delta)\right) + 2\delta nr(i-k)(P+\sqrt{Q})^{\frac{1}{3}} < 2^{\frac{2}{3}}(P+\sqrt{Q})^{\frac{2}{3}}$.

Proof.

1. Clearly $(0,0,0)$ is unstable.
2. At the axial equilibrium point $(k,0,0)$, eigenvalues of the Jacobian matrix are $\lambda_{11} = -r$, $\lambda_{12} = \beta k - \alpha$ and $\lambda_{13} = \delta - h$. For stability they must satisfy $k < \frac{\alpha}{\beta}$ and $\delta < h$.
3. Clearly predator extinct equilibrium $(\frac{\alpha}{\beta}, \frac{r(\beta k-\alpha)}{\beta^2 k}, 0)$ exist if $k > \frac{\alpha}{\beta}$. At this point Eigenvalues of the Jacobian matrix are

$\lambda_{21} = \delta - h$, $\lambda_{22} = \frac{-\alpha r+\sqrt{-4\alpha\beta^2k^2r+4\alpha^2\beta kr+\alpha^2r^2}}{2\beta k}$,

$\lambda_{23} = -\frac{\alpha r+\sqrt{-4\alpha\beta^2k^2r+4\alpha^2\beta kr+\alpha^2r^2}}{2\beta k}$. For stability we must have eigenvalues with negative real parts. For this the sufficient conditions are $\delta < h$ and $k > \frac{\alpha}{\beta}$.

4. For disease free euilibrium point $(u*, 0, w*)$: $u*$ is roots of the equation $u^*(\delta inr\,(a-k) + i\,k\,\lambda_1(\delta-h)) - a\,\delta\,i\,k\,n\,r + \delta\,n\,r(u^*)^2(i-k) + \delta\,n\,r(u^*)^3 = 0$ and $w^* = \frac{\delta u^*-hu^*}{\delta n}$. The nature of this equation assures that there exists atleast one positive root. For existence of w^* we must have $\delta > h$. One of the eigenvalues is $\lambda_{31} = \frac{-a\delta n(\alpha-\beta u^*)-\lambda_2 u^*(\delta-h)}{a\delta n}$. And other eigenvalues are roots of the the equation:

$$A\lambda^2+B\lambda+c=0 \quad (3)$$

where $A=\delta kn(ai+u(i+u))^2$

$B=\delta n(ai+u(i+u))^2(-hk+k(\delta-r)+2ru)-ik\lambda_1 u(h-\delta)\left(ai-u^2\right)$

$C=(h-\delta)\left(i^2k\lambda_1 u(2a+u)(h-\delta)+\delta nr(k-2u)(ai+u(i+u))^2\right)$

Applying Routh criteria for stability of equilibrium points in (4), since $A > 0$, the conditions $B > 0$ and $C > 0$ must be satisfied. Also $\lambda_{31} = \frac{-a\delta n(\alpha-\beta u)-\lambda_2 u(\delta-h)}{a\delta n} < 0$ must hold. 5. For the interior point of equilibrium (u^*, v^*, w^*), u^* is root of the equation

$$P_1u^7+P_2u^6+P_3u^5+P_4u^4+P_5u^3+P_6u^2+P_7u^+P_8$$

where $P_1=n^2r^2\beta\delta^2$

$P_2=-n^2r(ik\beta^2+r(\alpha+2(k-i)\beta)\delta^2$

$P_3=n\delta(n(r\beta(r-2k\beta)i^2+(ak^2\beta^3+kr(\alpha+k\beta)\beta-2r^2(\alpha-a\beta+2k\beta))i$
$+kr^2(2\alpha+k\beta))\delta+2ikr\beta(\delta-h)\lambda_1+ik\beta(k\beta-r)(h-\delta)\lambda_2)$

$P_4=n\delta(n(ai(-2(\alpha-i\beta+2k\beta)r^2-2ik\beta^2r+k^2\beta^2(2i\beta-\alpha))-r(k\beta^2i^3$
$+(r(\alpha+2k\beta)-2k\beta(\alpha+k\beta))i^2+k(k\alpha\beta-2r(2\alpha+k\beta))i+k^2r\alpha))\delta$
$+ik(ik\beta^2+r(2\alpha-i\beta+2k\beta))(h-\delta)\lambda_1+ik\beta(-2ir+kr+2ik\beta)(h-\delta)\lambda_2)$

$P_5=i(ik^2\beta(h-\delta)^2\lambda_1^2+k(h-\delta)(n(\beta(r+k\beta)i^2+(r(\alpha-2a\beta+k\beta)$
$-k\beta(\alpha+2a\beta))i-2kr\alpha)\delta+i(r-k\beta)(h-\delta)\lambda_2)\lambda_1+n\delta(n(i\beta(r^2+2k^2\beta^2)a^2$
$+(2k(2\alpha+k\beta)r^2+i^2k\beta^2(k\beta-2r)-2i((\alpha+2k\beta)r^2-k\beta(\alpha+k\beta)r$
$+k^2\alpha\beta^2))a+kr(\beta(\alpha+k\beta)i^2+(2r\alpha-2k\beta\alpha+kr\beta)i-2kr\alpha))\delta+ik\beta(-2ar-ir$
$+2kr+2ak\beta+ik\beta)(h-\delta)\lambda_2))$

$P_6=-i(ik^2(\alpha+i\beta)(h-\delta)^2\lambda_1^2-ik(h-\delta)((i(r-k\beta)-kr)(h-\delta)\lambda_2$
$-n(ir\alpha+kr\alpha+ik\beta\alpha+ikr\beta-a(r(2\alpha+i\beta+2k\beta)+k\beta(2\alpha-i\beta)))\delta)\lambda_1$
$+n\delta(n(i((\alpha+2k\beta)r^2+ik\beta^2r+2k^2\beta^2(\alpha-i\beta))a^2+$
$k(\beta(k\alpha\beta-2r(\alpha+k\beta))i^2-2r(2r\alpha-k\beta\alpha+kr\beta)i+2kr^2\alpha)a$
$+ik^2r\alpha(r+i\beta))\delta-ik\beta(ikr+2a(-ir+kr+ik\beta))(h-\delta)\lambda_2))$

$P_7=i^2k(ik(\alpha+a\beta)(h-\delta)^2\lambda_1^2-(h-\delta)(n(2a^2ik\beta^2-ikr\alpha$
$+a(ir\alpha+2kr\alpha-ik\beta\alpha+ikr\beta))\delta+i(-ar+kr+ak\beta)(h-\delta)\lambda_2)\lambda_1$
$+an\delta(n(a^2ik\beta^3-2kr\alpha(r+i\beta)+a((2\alpha+k\beta)r^2+i\beta(\alpha+k\beta)r$
$-2ik\alpha\beta^2))\delta+i\beta(-ar+2kr+ak\beta)(h-\delta)\lambda_2)$

$P_8 = -ai^2k^2\left(i\alpha(h-\delta)^2\lambda_1^2 + i(h-\delta)(r(h-\delta)\lambda_2 - n\alpha(r+2a\beta)\delta)\lambda_1 + an\delta\left(n\alpha\left(r^2+i\beta r+ai\beta^2\right)\delta + ir\beta(\delta-h)\lambda_2\right)\right)$

P_1 is positive. If $k >$ □ then $P_2 < 0$ and this ensures atleast one positive root to u^*.

Also $v^* = \frac{a\delta iknr-a\delta inru^*+hik\lambda_1u^*+\delta iknru^*-\delta ik\lambda_1u^*-\delta inr(u^*)^2+\delta knr(u^*)^2+\delta(-n)r(u^*)^3}{k(a\beta\delta in-hi\lambda_1+\delta i\lambda_1+\beta\delta inu^*+\beta\delta n(u^*)^2)}$

and $w^* = \frac{(\delta-h)(u^*+v^*)}{\delta n}$ For positive u^*, the change of signs in right hand side of v^* ensures atleast one positive value $v^*.w^*$ exists if $\delta > h$.

This assures the existence of internal equilibrium point.

V. GLOBAL STABILITY

Theorem 5.1 The point of equilibrium E=($\tilde{□}$,$\tilde{□}$,$\tilde{□}$) is globally stable if

$$\frac{\delta n}{(u+v)} < \frac{r}{k} - \lambda_1\frac{\tilde{□}(u+\tilde{□})}{a^2 i} - \frac{\lambda_2\tilde{□}(v+\tilde{□})}{a^2 i} - 2\tilde{□}$$

Proof. Global stability of the fixed point E = ($\tilde{□}$, $\tilde{□}$, $\tilde{□}$), may be analyzed by constructing a Lyapunov function in the following manner:

$$\Delta = w_1 + w_2 + w_3 \tag{4}$$

Where

$$w_1 = u - \square - \square \ln\left(\frac{u}{\square}\right), \quad w_2 = v - \square - \square \ln\left(\frac{v}{\square}\right),$$

$$w_3 = w - \square - \square \ln\left(\frac{w}{\square}\right)$$

Time derivative of equation 4 is

$$\frac{d\Delta}{dt} = \frac{dw_1}{dt} + \frac{dw_2}{dt} + \frac{dw_3}{dt} \tag{5}$$

where

$$\frac{dw_1}{dt} = \frac{u-\square}{u}\frac{du}{dt} = (u-\square)[r(1-\frac{u}{k}) - \frac{\lambda_1 w}{\frac{u^2}{i}+u+a} - \beta v]$$

$$= (\upsilon - \tilde{u})(\frac{-\rho}{\kappa}(\upsilon-\tilde{u}) + \frac{\lfloor_1 \tilde{w}}{\frac{\tilde{u}^2}{\iota}+\tilde{u}+\alpha} - \frac{\lfloor_1 \omega}{\frac{\upsilon^2}{\iota}+\upsilon+\alpha} - ®(\varpi - \tilde{v}))$$

$$= (u-\square)[\frac{-r}{k}(u-\square) - \beta(v-\square) + \frac{\lambda_1}{\left(\frac{\square^2}{i}+\square+a\right)\left(\frac{u^2}{i}+u+a\right)}(\frac{1}{i}\square(u+\square)(u-\square) - \frac{u^{*2}}{i}(w-\square) + \square(u-\square) - \square(w-\square) - a(w-\square))] \tag{6}$$

$$\frac{dw_2}{dt} = \frac{v-\square}{v}\frac{dv}{dt}$$
$$= (v-\square)[\beta(u-\square) + \frac{\lambda_2 \square}{\frac{\square^2}{i}+\square+a} - \frac{\lambda_2 w}{\frac{v^2}{i}+v+a}]$$
$$= (v-\square)[\beta(u-\square) + \frac{\lambda_2}{(\frac{\square^2}{i}+v+a)(\frac{v^2}{i}+v+a)}(\frac{1}{i}\square(v+\square)(v-\square) - \frac{\square^2}{i}(w-\square) + \square(v-\square) - \square(w-\square) - a(w-\square))] \tag{7}$$

$$\frac{dw_3}{dt} = \frac{w-\tilde{w}}{w}\frac{dw}{dt}$$
$$= \delta n[\frac{-(w-\tilde{w})^2}{u+v} + \frac{\tilde{w}(w-\tilde{w})(u-\tilde{u})}{(u+v)(\tilde{u}+\tilde{v})} + \frac{\tilde{w}(w-\tilde{w})(v-\tilde{v})}{(u+v)(\tilde{u}+\tilde{v})}] \tag{8}$$

Substituting (7), (8), (9) in equation (6) we get

$$\frac{d\Delta}{dt} = (u-\tilde{u})^2[\frac{-r}{k} + \frac{\lambda_1}{(\frac{\tilde{u}^2}{i}+\tilde{u}+a)(\frac{u^2}{i}+u+a)}(\frac{\tilde{w}(u+\tilde{u})}{i}+\tilde{w}))]$$
$$+(u-\tilde{u})(w-\tilde{w})[\frac{\lambda_1}{(\frac{\tilde{u}^2}{i}+\tilde{u}+a)(\frac{u^2}{i}+u+a)}(\frac{-\tilde{u}^2}{i}-\tilde{u}-a) + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})}]$$
$$+(v-\tilde{v})^2[\frac{\lambda_2}{(\frac{\tilde{v}^2}{i}+\tilde{v}+a)(\frac{v^2}{i}+v+a)}(\frac{\tilde{w}(v+\tilde{v})}{i}+\tilde{w})]$$
$$+(v-\tilde{v})(w-\tilde{w})[\frac{\lambda_2}{(\frac{\tilde{v}^2}{i}+\tilde{v}+a)(\frac{v^2}{i}+v+a)}(\frac{-\tilde{v}^2}{i}-\tilde{v}-a+\frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})})]$$
$$+(w-\tilde{w})^2[\frac{-\delta n}{u+v}] \tag{9}$$

$$\frac{d\Delta}{dt} \le (u-\tilde{u})^2[\frac{-r}{k} + \frac{\lambda_1}{a^2}(\frac{\tilde{w}(u+\tilde{u})}{i}+\tilde{w}) + \frac{1}{2}(\frac{-\lambda_1}{(\frac{u^2}{i}+u+a)} + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})})]$$
$$+(v-\tilde{v}^2)[\frac{\lambda_2}{a^2}(\frac{\tilde{w}(v+\tilde{v})}{i}+\tilde{w}) + \frac{1}{2}(-\frac{\lambda_2}{\frac{v^2}{i}+v+a} + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})})]$$
$$+\frac{(w-\tilde{w})^2}{2}[\frac{-\lambda_1}{(\frac{u^2}{i}+u+a)} + \frac{\delta n w^*}{(u+v)(u^*+v^*)} - \frac{\lambda_2}{\frac{v^2}{i}+v+a} + \frac{\delta n w^*}{(u+v)(u^*+v^*)} - \frac{2\delta n}{u+v}]$$

For negative definite the following conditions must be satisfied:

$$\frac{-r}{k} + \frac{\lambda_1}{a^2}(\frac{\tilde{w}(u+\tilde{u})}{i}+\tilde{w}) + \frac{1}{2}(\frac{-\lambda_1}{(\frac{u^2}{i}+u+a)} + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})}) < 0 \tag{10}$$

$$\frac{\lambda_2}{a^2}(\frac{\tilde{w}(v+\tilde{v})}{i}+\tilde{w}) + \frac{1}{2}(-\frac{\lambda_2}{\frac{v^2}{i}+v+a} + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})}) < 0 \tag{11}$$

$$\frac{-\lambda_1}{(\frac{u^2}{i}+u+a)} + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})} - \frac{\lambda_2}{\frac{v^2}{i}+v+a} + \frac{\delta n\tilde{w}}{(u+v)(\tilde{u}+\tilde{v})} - \frac{2\delta n}{u+v} < 0 \tag{12}$$

From (11) (12) and (13) we obtain

$$\frac{\delta n}{(u+v)} < \frac{r}{k} - \lambda_1\frac{\tilde{w}(u+\tilde{u})}{a^2 i} - \frac{\lambda_2 w\,\tilde{w}(v+\tilde{v})}{a^2 i} - 2\tilde{w}$$

VI. NUMERICAL ANALYSIS

Numerical solution of the model 2 is obtained by using Runge-Kutta method for various sets of parameter values to verify the stability of equilibrium points.

Axial Equilibrium: The parameters values are given as: $r = 2; k = 50; \lambda_1 = 0.1; \beta = 0.003; \lambda_2 = 0.29; \alpha = 0.2; \quad \delta = 0.5; n = 2; h = 0.7; i = 0.1; a = 1;$. They satisfy the condition $k < \frac{\alpha}{\beta} = 66.6667$ and $\delta < h$. At the equilibrium point is $E_1(2,0,0)$ eigenvalues of the Jacobian matrix are $\{-2., -0.2, -0.05\}$. The equilibrium point $E_1(50,0,0)$ is a stable and it is presented in Fig.1

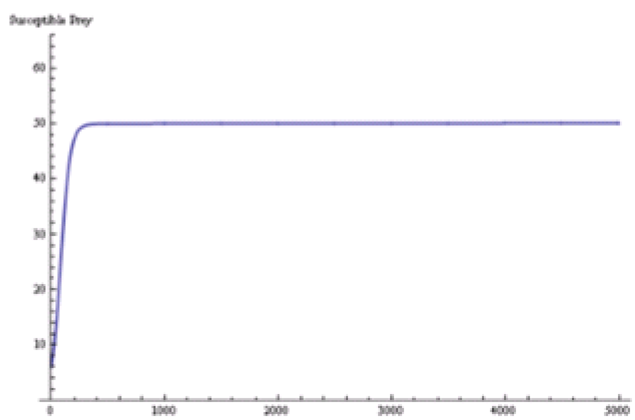

Fig. 1

Predator Extinct Equilibrium: Parameters values are taken as $r = 0.2; k = 100; \lambda_1 = 0.1; \beta = 0.001; \lambda_2 = 0.29; \alpha = 0.05; \delta = 0.3; n = 2; h = 0.5; i = 0.3; a = 0.3; n = 2; h = 0.5; i = 0.3; a = 1;$ These values satisfy the required condition $k > \frac{\alpha}{\beta} = 50$ and $\delta < h$. Eigenvalues of Jacobian matrix at point of Equilibrium $E_2(50,100,0)$ are $\{-14.7944, -0.405558, -0.2\}$. Therefore E_2 is a stable focus (Fig. 2).

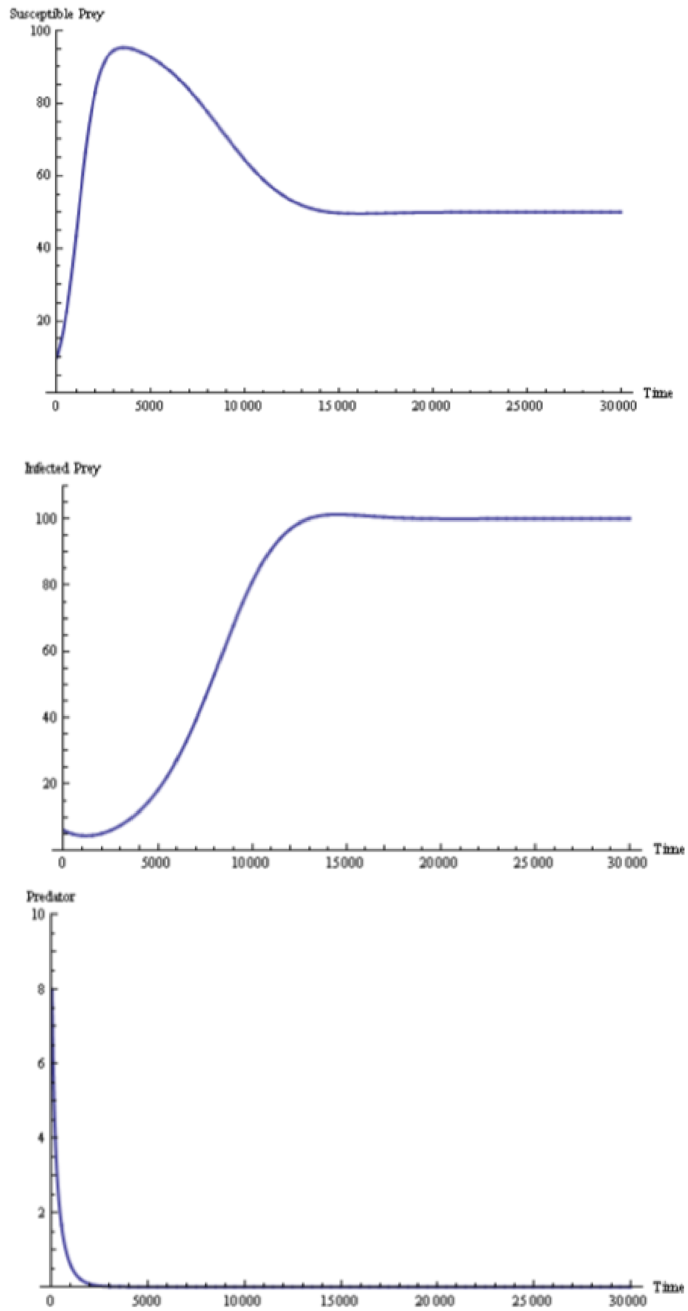

Fig. 2

Disease Free Equilibrium: Parameters values are taken as $r = 0.2; k = 100; \lambda_1 = 0.1; \beta = 0.003; \lambda_2 = 0.29; \alpha = 0.1; \delta = 0.5; n = 1; h = 0.3; i = 0.1; a = 1$. These values satisfy the required condition $\delta > h$ and $A > 0$, $B > 0$ and $C > 0$. Eigenvalues of Jacobian matrix at point of Equilibrium $E_3(99.96, 0w, 39.984)$ are $\{-11.3955, -0.19992, -0.09996\}$. Therefore E_3 is a stable focus(Fig. 3).

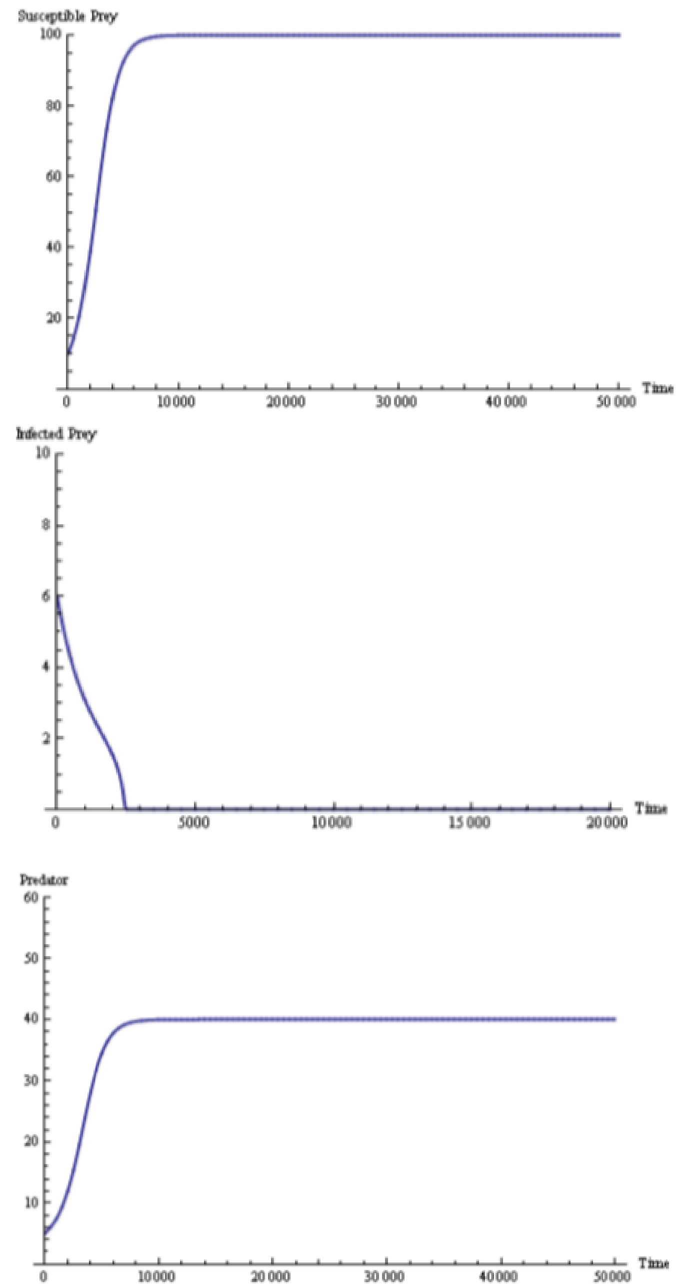

Fig.3

Internal Equilibrium: Parameters values are taken as $r = 4; k = 100; \lambda_1 = 0.1; \beta = 0.3; \lambda_2 = 0.29; \alpha = 1; \delta = 5; n = 6; h = 0.5; i = 0.4; a = 3$. These values satisfy the required condition for equilibrium. Eigenvalues of Jacobian matrix at point of Equilibrium $E_4(3.3388, 12.8645, 2.43049)$ are $\{-4.49647, -0.0548444 + 1.96844i, -0.0548444 - 1.96844i\}$. Therefore E_4 is a stable spiral(Fig.4).

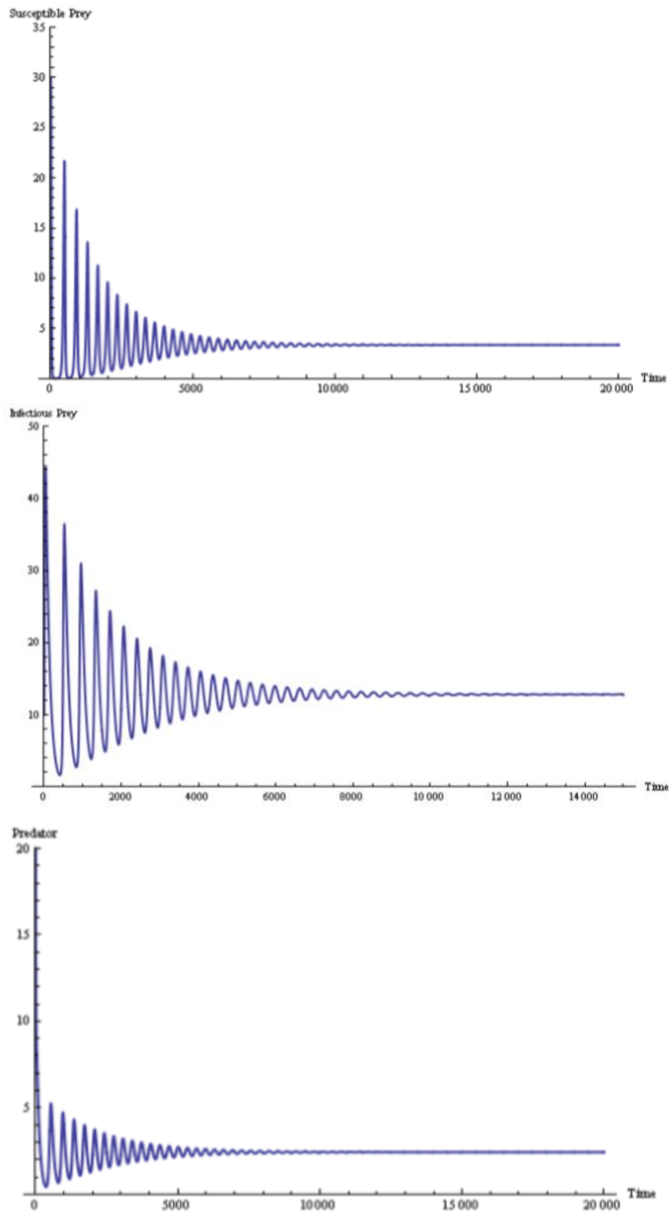

Fig.4

VII. CONCLUSION

A L-G predator-prey model proposed in the form of the model 2 which has Holling type-IV FR and disease term in prey in the presence of predator harvesting, is analyzed in this paperl. Conditions, derived for uniform Boundedness and positivity of the solutions, established the necessity of them. The proposed system has trivial, axial, predator extinct, disease free and interior equilibrium points. From the conditions derived for existence and stability of points of equilibrium, it is observed that existence and stability of the coexistence equilibrium point depends on predator immunity i, growth rate δ and predator harvesting. Also stability of disease free equilibrium and predator extinct equilibrium donot coexist. But, disease free equilibrium point and internal equilibrium point always coexist. Therefore, in the model 2, the prey population tends either to a disease free state where it tends to the carrying capacity or to a endemic state. Numerical results projected in Table 2 to Table 5 shows good support to the analytical results.

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