Introductory Note On Affine Transformation In Mathematics

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Abstract— Every branch of science and social science deals with specific sets of elements and the structure of relationship between these elements. When we want to study this structure, it is useful to study the various possible transformations on the elements. Also it is useful to determine which quantities and properties remain invariant under these transformations. Clearly invariance depends on the set of transformations under consideration. We know that mass is an invariant for Newtonian, but not for relativistic transformations; total linear momentum of a given system of particles is invariant for transformations that do not involve external forces, but is not necessarily so for transformations that involve external forces.

Key Words— Affine, Group, Mathematics, Model, Transformation, Structure, System .

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1 Introduction

Mathematics provides models having same structure those in social, physical and biological sciences. It also deals with sets, structures on these sets, transformations in them and their invariants. Clearly mathematics deals with relatively more abstract sets and structures than other sciences. Mathematics essentially deals with three basic or mother structures, viz., algebraic, topological, and order, and most mathematical systems exhibit these structures singly or fused together [1] .

The central idea in modern mathematics is that the study of a structure is best made in terms of transformations which preserve that structure. In fact the application of mathematics to nature and society is also based on this idea [1].All sciences deal with structure and in applying mathematics to them we attempt to find mathematical systems having structures similar to those in physical, biological and social sciences.

A physical, biological or social situation may be highly complicated, but at one time we are interested only in some particular aspect of it. We first find an appropriate transformation to convert this situation into a mathematical form which preserves the structure we are interested in. This process is called mathematical model making. The mathematical model is as good or as bad as its capacity to preserve the particular aspect of the structure we are considering. So our problem is to find a transformation which is as close to reality as possible and which is capable of being handled by appropriate simplifying mathematical transformations. Thus mathematical model making is an exercise in finding suitable physical mathematical transformations preserving the structures under study, keeping in mind the available mathematical transformations for simplifying the mathematical model. Once we have formed a mathematical model, we proceed to find mathematical transformations which will reduce it to a simpler form and preserve the structure in question or keep a certain property invariant.

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Group of Transformations

Transformation is a one-one, mapping of a set into itself. Two transformations S and T are said to be equal if and only if they are defined on the same set X and are such that $Sx = Tx \ \forall\, x \in X$.

Now let G be the set of all transformations on X. Let T_1, T_2, T_3 be any three transformations of the set. If $T_1: x \to y$ and $T_2: y \to z$, then the mapping which maps x to z is called the product of these two transformations and we write it as T_2T_1, so that $T_2T_1: x \to z$, so that T_2T_1, determines a unique image z for every element $x \in X$, and every element z of X has a unique pre image x by T_2T_1. Thus the product of two transformations is also a transformation so we can say that closure property holds in G.

i. Also if $T_3: z \to$u

Then $[(T_3T_2)T_1]x = (T_3T_2)T_1x = (T_3T_2)y = T_3(T_2y) = T_3z = u$,

$$\& \ [T_3(T_2T_1)]x = T_3[(T_2T_1)x] = T_3[T_2(T_1x)] = T_3(T_2y) = T_3z = u$$

So for every $T_1, T_2, T_3 \in G$, we have $(T_3T_2)T_1 = T_3(T_2T_1)$

∴Thus associative law holds for multiplication of transformations

ii. The Transformation $I: x \to x$, is such that for every $T \in G$, we have

$(IT)x = I(Tx) = Iy = y = Tx$,

& $(TI)x = T(Ix) = Tx = y = Tx$,

So that $IT = TI = T$,

Thus identity transformation $I \in G$.

iii.

For every transformation $T: x \to y$ in G, there exists a mapping T^{-1} defined by $T^{-1}: y \to x \ in\ G$, which is one-one, onto and s such that $(TT^{-1})y = T(T^{-1}y) = Tx = y = Iy$,

$$\& \ (T^{-1}T)x = T^{-1}(Tx) = T^{-1}y = x = Ix$$

$$\Rightarrow TT^{-1} = T^{-1}T = I$$

∴ Thus every transformation in G has an inverse transformation in G.

Since all postulates are satisfied in G, so set of all transformations on X forms a group for the operation of multiplication of transformations. Group G is not commutative in general. In geometry, the groups of greatest interest are those which arise when X is the Euclidean space of n dimensions, where $n = 1,2,3,4,5, \dots$ ..... Group of transformations which are likely to be of interest are those which preserve some geometrical property such as incidence, co linearity, concurrence, parallelism, distance, order, angle, direction, area, co circularity, congruence, similarity, continuity, cross ratio of points etc. Most of the modern curriculum reform projects in school mathematics [2] have given an important place to transformation geometry, especially to the study of isometric (distance preserving) transformations. Several new undergraduate courses also contain a discussion of metric geometry over affine space [3, 4].

Affine Transformations

Consider the mapping $x' = ax + by + d,\ y' = lx + my + n$ ……..(i) from the set of all ordered pairs (x, y) of real numbers to the set of ordered pairs $(x',\ y')$ of real numbers. So (1) is a mapping from the xy plane onto itself. For this mapping to be one-one, we should have a unique (x, y) to correspond to a given (x', y'). Solving (1) for x and y, we get

$$x = \frac{m}{am-bl}x' - \frac{b}{am-bl}y' + \frac{bn-dm}{am-bl}$$

and

$$y = -\frac{l}{am-bl}x' + \frac{a}{am-bl}y' + \frac{dl-an}{am-bl} \dots \text{(ii)}$$

Thus, to a given point$(x',\ y')$, there corresponds a unique point (x, y) if $am - bl \neq 0 \dots \dots (iii)$

Therefore (1) gives a one-one mapping of the Cartesian plane onto itself if (3) is satisfied. This transformation is called an affine transformation and the quantity $am - bl = \begin{vmatrix} a & b \\ l & m \end{vmatrix}$ is known as the determinant or the transformation.

Hence, an affine transformation is given by

$$x' = ax + by + d,\ y' = lx + my + n,\ \ am - bl \neq 0 \dots \dots \dots (iv)$$

Now, (ii) will determine an affine transformation if

$$\left(\frac{m}{am-bl}\right)\left(\frac{a}{am-bl}\right) - \left(-\frac{b}{am-bl}\right)\left(-\frac{l}{am-bl}\right) \neq 0$$

$i.e., if\ (am - bl)^{-1} \neq 0,$ which is satisfied because of (iii). Equations (ii) also determine an affine transformation.

If transformation (i) sends a point M to point N , then transformation (ii) sends point N back to M. Transformation (ii) is called the transformation inverse to transformation (i). Hence, every affine transformation has an inverse transformation which itself is an affine transformation. Determinant of the inverse transformation is the reciprocal of the reciprocal of the determinant of the original transformation.

The product of two transformations

$$x' = ax + by + d,\ y' = lx + my + n,$$

$$and\ x'' = \frac{m}{am-bl}x' - \frac{b}{am-bl}y' + \frac{bn-dm}{am-bl}$$

$$y'' = -\frac{l}{am-bl}x' + \frac{a}{am-bl}y' + \frac{dl-an}{am-bl}$$

$where\ am - bl \neq 0$ is given by

$$x'' = \frac{m}{am-bl}(ax+by+d) - \frac{b}{am-bl}(lx+my+n) + \frac{bn-dm}{am-bl} = x,$$ and

$$y'' = -\frac{l}{am-bl}(ax+by+d) + \frac{a}{am-bl}(lx+my+n) + \frac{dl-an}{am-bl} = y$$

This is the identity transformation and is also affine transformation.

∴ Each of the transformation (i) and (ii) is the inverse of the other.

Now we show that the set of all affine transformation forms a group.-

1. The product of two affine transformations
   $x' = ax + by + d,\ y' = lx + my + n,\ \ am - bl \neq 0 \dots \dots \dots$ (i) and
   $x'' = a'x' + b'y' + d',\ y'' = l'x' + m'y' + n',\ \ a'm' - b'l' \neq 0 \dots \dots \dots$ (ii) is given by
   $x'' = a'(ax + by + d) + b'(lx + my + n) + d'$,
   $y'' = l'(ax + by + d) + m'(lx + my + n) + n'$
   i.e. $x'' = (a'a + b'l)x + (a'b + b'm)y + a'd + b'n + d'$,
   $y'' = (l'a + m'l)x + (l'b + m'm)y + l'd + m'n + n'$,……………………………..(iii)
   This will be an affine transformation if
   $(a'a + b'l)(l'b + m'm) - (a'b + b'm)(l'a + m'l) \neq 0$
   i.e., if $(am - bl)(a'm' - b'l') \neq 0$
   This condition is satisfied due to (i) and (ii).
   Thus, the product of two affine transformations is an affine transformation. Closure property holds in the set of affine transformations for the operation of multiplication of transformations.
2. Let first transformation of (1) be called f and the second g.Then, transformation (iii) is denoted by gf.
   Let h be a third transformation defined by
   $x''' = a''x'' + b''y'' + d'',\ y''' = l''x'' + m''y'' + n'';\ \ a''m'' - b''l'' \neq 0 \dots \dots \dots$ (ii)
   Then, we can easily verify that $h(gf) = (hg)f$
   ∴Associative law holds for multiplication of transformation.
3. There exists an affine transformation I ,
   $x' = x, y' = y,$

We can easily see that for every affine transformation f,

$$fI = If = f$$

∴Set of affine transformation possesses an identity element.

1. For every affine transformation f, there exists an inverse affine transformation f^{-1} which is such that $ff^{-1} = f^{-1}f = I$
   Thus, the set of all affine transformation forms a group.

Properties of Affine Transformations

In this section some properties of affine transformation are discussed

P (1)- An affine transformation maps straight lines onto straight lines.

P (2)-An straight line maps a family of parallel straight lines onto another family of parallel straight lines.

P (3)-In general affine transformations do not preserve distances between points. Affine transformations which preserve distances are called isometric transformations.

P(4)- An affine transformation maps a triangle ABC onto a triangle $A'B'C'$, the points inside (outside) triangle ABC are mapped onto points inside (outside) triangle $A'B'C'$ and the centroid of triangle ABC is mapped onto the centroid of triangle $A'B'C'$.

P (5)- An affine transformation maps a bounded region onto a bounded region.

P (6)- An affine transformation maps a convex region onto a convex region.

P (7)- An affine transformation maps a parallelogram onto a parallelogram but does not necessarily map a square or a rectangle onto a square or a rectangle.

P (8)- An affine transformation maps a triangle ABC of area S onto a triangle $A'B'C'$ of area $S \times |am - bl|$.

P (9)- An affine transformation changes the area of all bounded regions in the plane in the same ratio 1: $|(am - bl)|$.

If$(am - bl) > 0$, orientations of areas do not change and the transformation is called a direct or proper affine transformation (D.A.T.).

If $(am - bl) < 0$, orientations of areas change and the transformation is called an indirect or improper or opposite affine transformation (O.A,T.).

If$(am - bl) = 1$, areas change neither in magnitude nor in orientation and the transformation is called a direct equiaffine transformation.

P (10)- The set of all direct affine transformations is a group.

P (11) –The set of all opposite affine transformations is not a group since the product of two opposite affine transformations is a direct affine transformation and the identity transformation does not belong to the set.

P (12)- The group of all direct affine transformations is a normal subgroup of all affine transformations.

P (13)- An affine transformation maps the centroid of a given region onto the centroid of the image region.

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