Numerical Solutions Of Fuzzy Multiple Hybrid Single Neutral Delay Differential Equations

D. Prasantha Bharathi, T. Jayakumar, S. Vinoth.

Abstract—In this article, the concept of hybrid system especially multiple hybrid system, differential system of neutral type are bound together with the concept of fuzzy. The Runge-Kutta method of order 4 is used as a tool to solve the problem numerically. Two problems are solved to verify the theoretical results. The exact solutions are obtained in a usual way of solving delay differential equations. The exact and numerical solutions are compared in the case of non-fuzzy and fuzzy valued solutions.

Index Terms— Delay Differential System, Hybrid differential equation, Multiple hybrid differential equation, Neutral Delay Differential system, Numerical Solutions, Runge-Kutta method, Time depended delay.

1 INTRODUCTION

The concept of fuzzy was introduced by Zadeh and later Dubois and Prade applied the idea of extension principle. Throughout the paper multiple hybrid system is bound with the concept of neutral delay differential equation and then we extend the concept with the idea of fuzzy differential equations. The Practical applications of hybrid systems are nowadays seems to exercise a lead role in communication ,engineering, etc., and neutral delay differential equation was one type of rare concept of delay differential equations, because its practical applications are not well established as like that of delay differential equations. This paper just extends the neutral differential equations to the next storey called multiple hybrid single neutral type of differential equations. We assume this system of differential equation as fuzzy multiple hybrid single neutral delay differential equations (FMHSNDDE). The essential theory is presented with governing equation and explained it with few examples. In [1] Alfredo Bellan and Marino Zennaro clearly explained Numerical methods for delay differential equations. Jayakumar, Parivallal and Prasantha Bharathi in [6] have treated fuzzy delay differential equation numerically. D. Prasantha Bharathi, T. Jayakumar and S. Vinoth used this type of Runge-Kutta method to solve various types of delay differential equations [10, 11, 12, 13 and 14]. Pederson investigated with Sambandham [9] the numerical solution of fuzzy hybrid differential equation by using Runge-Kutta method. Al-Rawi and others introduced in [14] the numerical method for solving Delay differential equations by RK-4.

In the below section 2, 3, 4, 5 and 6 the preliminaries, mathematical formulation of the problem, analytical and numerical solutions, their examples and conclusions are discussed respectively.

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2 PRELIMARIES

Throughout the paper, we are following some new abbreviations, Notations, etc., FMHSNDDE represents Fuzzy multiple hybrid Single Neutral Delay Differential equations. "~"represents fuzzy functions. RK-4 represents the Fourth order Runge-Kutta method. For the definitions of Fuzzy sets, Fuzzy differential equations, Hybrid fuzzy differential equations, Fuzzy delay differential equations, the readers may refer [1, 2, 7, 10, and 11].

3 MATHEMATICAL FORMULATION

With the knowledge acquired from [1, 7, 10, 11, 12 and 13], We are assuming the existence of new system called FMHSNDDE. Such system is modeled here as

$$\begin{cases} \tilde{y}'(t) = \min,\ \max\ f(t, y(y), y'(t-\tau), N\lambda(y_k(t_k))),\ t \geq t_0 \\ \tilde{y}(t) = \min,\ \max\ \phi(t), & -\tau \leq t \leq t_0 \\ \tilde{y}(t_0) = \min,\ \max\ y_0 \in \phi(t) \end{cases} \quad (1)$$

$f: [0,\infty) \times R \times R \times R \to R$, N is a natural number, used to denote the multiple of hybrid term and that was given by

$$\lambda(y_k(t_k)) = \begin{cases} 0, t_k = 0 \\ y_k, t_k > 0 \end{cases}, \text{Where } t_k = k, k = 0,1,2, \dots\ y(t_k) = y_k.$$

Since $\tilde{y}$ represents the fuzzy type of solutions of y, Each solution was multiplied by minimum and maximum fuzzy numbers such that the values coincide at the maximum value of r, where r varies in [0,1]. $\tau \in R$, is a delay term such that $y'(t-\tau)$ represents the neutral delay term. This term is the reason for calling (1) to be neutral differential equations. Since both hybrid term and neutral delay term are set together, (1) is so called as FMHSNDDE.

4 ANALYTICAL SOLUTION AND NUMERICAL SOLUTION

Alfredo Bellan et.al.,[1] converted neutral delay to delay differential equations. But here the analytical solutions are obtained by usual integration technique meant for solving the delay differential equations. Since it includes neutral delay term, we have to integrate it with the limits so that the new value obtained will become the new initial functions for the next iteration. For each interval the hybrid term remains the same but the delay term will change for each interval.

The numerical solutions are obtained by using the RK-4 method algorithm which is as follows.

$$\tilde{K}_1 = \min,\ \max(\ hf\ (t, y(t), y'(t-\tau), N\lambda(y_k(t_k))))$$
$$\tilde{K}_2 = \min,\ \max(\ hf\ (t+h/2, y(t)+\tilde{K}_1/2, y'(t-\tau)+\tilde{K}_1/2, N\lambda(y_k(t_k)+\tilde{K}_1/2)))$$
$$\tilde{K}_3 = \min,\ \max(\ hf\ (t+h/2, y(t)+\tilde{K}_2/2, y'(t-\tau)+\tilde{K}_2/2, N\lambda(y_k(t_k)+K_2/2)))$$
$$\tilde{K}_4 = \min,\ \max(\ hf\ (t+h, y(t)+\tilde{K}_3, y'(t-\tau)+\tilde{K}_3, N\lambda(y_k(t_k)+K_3)))$$
$$\tilde{y}_{n+1} = \tilde{y}_n + h/6.(\tilde{K}_1 + 2\tilde{K}_2 + 2\tilde{K}_3 + \tilde{K}_4)$$

The $y'(t-\tau)$ is solved integrated separately for each iteration. If it is the function of t, h terms in RK-4 can be added else it is the function of $y(t), K_i$ terms should be added and it should be followed for each iteration of $t \in [t_0, t_n]$.

5 NUMERICAL EXAMPLES

In this section we choose to solve two problems. For easy understanding of the procedures we recommend readers to follow [1,7,11]. For same delay term and different multiple of hybrid terms we are going to observe the fuzzy type solutions in the following two problems.

Example 5.1

$$\begin{cases} \tilde{y}'(t) = (0.75 + 0.25r, 1.125 - 0.125r) \\ \quad y'(t-1) + 2m(t)\lambda_k y(t_k), & 0 \le t \le 3 \\ \tilde{y}(t) = (0.75 + 0.25r, 1.125 - 0.125r)e^t, & -1 \le t \le 0 \\ m(t) = \left|Sin(\pi t)\right|, & 0 \le t \le 3 \end{cases}$$

The analytically obtained solutions are given as
$\tilde{Y}(t) = (0.75 + 0.25r, 1.125 - 0.125r)$.

$$\begin{cases} e^t & t < 0 \\ -\dfrac{-2e^2 + \pi - e\pi - e^t\pi + 2e^2 Cos[\pi t]}{e\pi} & 0 \le t < 1 \\ \dfrac{4e^3 - 2e\pi + 2e^2\pi + e^t\pi}{e^2\pi} & 1 \le t < 2 \\ -\dfrac{-6e^4 + 3e^2\pi - 3e^3\pi - e^t\pi + 2e^4 Cos[\pi t]}{e^3\pi} & 2 \le t \le 3 \end{cases}$$

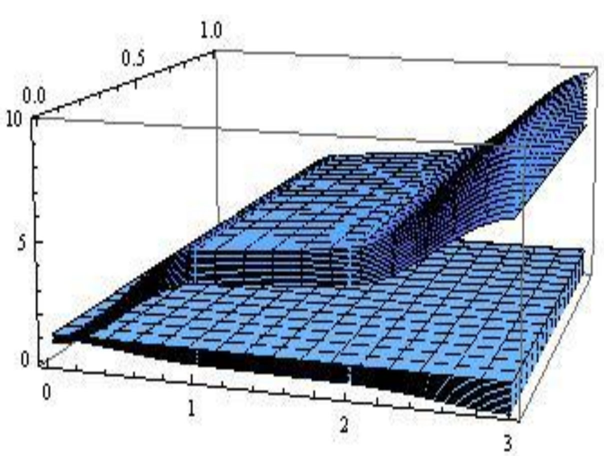

Figure:5.1.1 Fuzzy type solutions for $t \in [0,3], r \in [0,1]$.

The approximately obtained solutions are compared with that of analytically obtained solutions in the following table for both non- fuzzy and fuzzy valued solutions.

Table: 5.1.1: *Comparision of Non- Fuzzy Solutions*

t.	Analytical Solution	Approximate Solution	Error Analysis
0	1	1	0.00000
0.1	1.12339	1.12339	0.00000
0.2	1.41195	1.41195	0.00000
0.3	1.84205	1.84205	0.00000
0.4	2.37669	2.37669	0.00000
0.5	2.96916	2.96917	0.00001
0.6	3.56771	3.56772	0.00001
0.7	4.12062	4.12063	0.00001
0.8	4.58138	4.58139	0.00001
0.9	4.91328	4.9133	0.00002
1.0	5.09314	5.09316	0.00002
1.1	5.13183	5.13185	0.00002
1.2	5.17459	5.17461	0.00002
1.3	5.22185	5.22186	0.00001
1.4	5.27408	5.27409	0.00001
1.5	5.3318	5.33181	0.00001
1.6	5.39559	5.3956	0.00001
1.7	5.46608	5.4661	0.00002
1.8	5.544	5.54401	0.00001
1.9	5.6301	5.63011	0.00001
2.0	5.72527	5.72528	0.00001
2.1	5.84865	5.84866	0.00001
2.2	6.13721	6.13723	0.00002
2.3	6.56731	6.56733	0.00002
2.4	7.10195	7.10197	0.00002
2.5	7.69443	7.69445	0.00002
2.6	8.29298	8.29299	0.00001
2.7	8.84589	8.84591	0.00002
2.8	9.30664	9.30666	0.00002
2.9	9.63855	9.63857	0.00002
3.0	9.81841	9.81843	0.00002

Table: 5.1.2:Comparision of Fuzzy Solutions

t=3	Analytical Solution		Approximate Solution	
r.	Minimum	Maximum	Minimum	Maximum
0	7.36381	11.0457	7.36382	11.0457
0.1	7.60927	10.923	7.60929	10.923
0.2	7.85473	10.8003	7.85475	10.8003
0.3	8.10019	10.6775	8.10021	10.6775
0.4	8.34565	10.5548	8.34567	10.5548
0.5	8.59111	10.4321	8.59113	10.4321
0.6	8.83657	10.3093	8.83659	10.3094
0.7	9.08203	10.1866	9.08205	10.1866
0.8	9.32749	10.0639	9.32751	10.0639
0.9	9.57295	9.94114	9.57297	9.94116
1.0	9.81841	9.81841	9.81843	9.81843

***Table: 5.1.3:** Error Analysis Fuzzy Solutions*

t=3	Error Analysis	
r.	Minimum	Maximum
0	0.00001	0.00000
0.1	0.00002	0.00000
0.2	0.00002	0.00000
0.3	0.00002	0.00000
0.4	0.00002	0.00000
0.5	0.00002	0.00000
0.6	0.00002	0.00010
0.7	0.00002	0.00000
0.8	0.00002	0.00000
0.9	0.00002	0.00002
1.0	0.00002	0.00002

Example 5.2

$$\begin{cases} \tilde{y}'(t) = (0.75 + 0.25r, 1.125 - 0.125r) \\ \quad y'(t-1) + 3m(t)\lambda_k y(t_k), & 0 \le t \le 3 \\ \tilde{y}(t) = (0.75 + 0.25r, 1.125 - 0.125r)e^t, & -1 \le t \le 0 \\ m(t) = |Sin(\pi t)|, & 0 \le t \le 3 \end{cases}$$

The analytically obtained solutions are given as

$$\tilde{Y}(t) = (0.75 + 0.25r, 1.125 - 0.125r).$$

$$\begin{cases} e^t & t < 0 \\ -\dfrac{-3e^2 + \pi - e\pi - e^t\pi + 3e^2 Cos[\pi t]}{e\pi} & 0 \le t < 1 \\ \dfrac{6e^3 - 2e\pi + 2e^2\pi + e^t\pi}{e^2\pi} & 1 \le t < 2 \\ -\dfrac{-9e^4 + 3e^2\pi - 3e^3\pi - e^t\pi + 3e^4 Cos[\pi t]}{e^3\pi} & 2 \le t \le 3 \end{cases}$$

The two dimensional fuzzy valued analytical solutions are plotted as

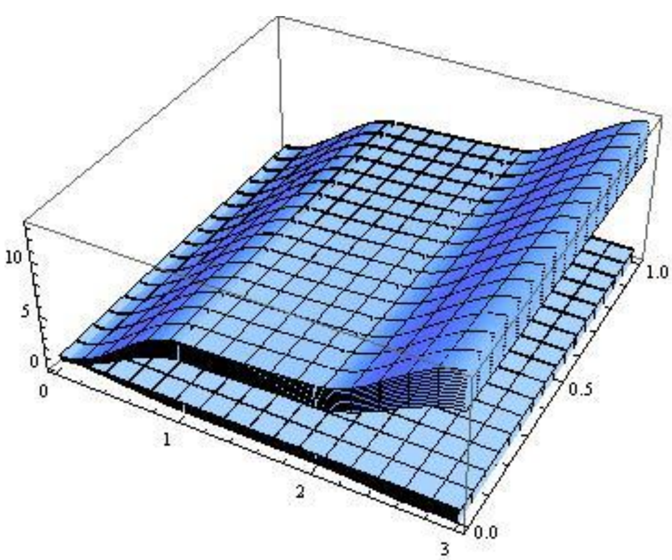

***Figure:5.2.1** Fuzzy type solutions for $t \in [0,3], r \in [0,1]$.*

Table: 5.2.1: Comparision of Non- Fuzzy Solutions

t.	Analytical Solution	Approximate Solution	Error Analysis
0	1	1	0.00000
0.1	1.16574	1.16574	0.00000
0.2	1.5772	1.5772	0.00000
0.3	2.19872	2.19872	0.00000
0.4	2.97456	2.97457	0.00001
0.5	3.83442	3.83443	0.00001
0.6	4.70034	4.70036	0.00002
0.7	5.49446	5.49447	0.00001
0.8	6.14664	6.14666	0.00002
0.9	6.60145	6.60147	0.00002
1.0	6.82366	6.82367	0.00001
1.1	6.86235	6.86236	0.00001
1.2	6.90511	6.90512	0.00001
1.3	6.95236	6.95238	0.00002
1.4	7.00459	7.00461	0.00002
1.5	7.06231	7.06233	0.00002
1.6	7.1261	7.12611	0.00001
1.7	7.1966	7.19661	0.00001
1.8	7.27451	7.27453	0.00002
1.9	7.36061	7.36063	0.00002
2.0	7.45578	7.45579	0.00001
2.1	7.62151	7.62153	0.00002
2.2	8.03297	8.03299	0.00002
2.3	8.6545	8.65452	0.00002
2.4	9.43034	9.43036	0.00002
2.5	10.2902	10.2902	0.00000
2.6	11.1561	11.1562	0.00010
2.7	11.9502	11.9503	0.00010
2.8	12.6024	12.6025	0.00010
2.9	13.0572	13.0573	0.00010
3.0	13.2794	13.2795	0.00010

Table:5.2.2:Comparision of Fuzzy Solutions

t.	Analytical Solution		Approximate Solution	
r.	Min	Max	Min	Max
0	9.95958	14.9394	9.9596	14.9394
0.1	10.2916	14.7734	10.2916	14.7734
0.2	10.6235	14.6074	10.6236	14.6074
0.3	10.9555	14.4414	10.9556	14.4414
0.4	11.2875	14.2754	11.2875	14.2754
0.5	11.6195	14.1094	11.6195	14.1094
0.6	11.9515	13.9434	11.9515	13.9434
0.7	12.2835	13.7774	12.2835	13.7774
0.8	12.6155	13.6114	12.6155	13.6115
0.9	12.9474	13.4454	12.9475	13.4455
1.0	13.2794	13.2794	13.2795	13.2795

TABLE: 5.2.3: ERROR ANALYSIS FUZZY SOLUTIONS

t=3	Error Analysis	
r.	Minimum	Maximum
0	0.00002	0.00000
0.1	0.00000	0.00000
0.2	0.00010	0.00000
0.3	0.00010	0.00000
0.4	0.00000	0.00000
0.5	0.00000	0.00000
0.6	0.00000	0.00000
0.7	0.00000	0.00000
0.8	0.00000	0.00010
0.9	0.00010	0.00010
1.0	0.00010	0.00010

6 CONCLUSION

Two problems were solved for different values of N, Same values of τ and $m(t)$. The tables were given and their respective fuzzy plots were also given. By taking the fuzzy values to all the values of t Similar to Table:5.1.2 and Table:5.2.2 , the fuzzy plots, Figure 5.1.1 and 5.2.1 were obtained. From the comparison of Fuzzy approximate values and Fuzzy analytical values it was clear that they coincide well. The error analysis of the non-fuzzy values and fuzzy also reveals that the solutions obtained by RK-4 is coinciding well with exact solutions. Thus in the case difficulty in finding analytical solutions of any complicated FMHSNDDE, Runge-Kutta method of order four can be recommended. In future Semi analytical methods are also used and then compared with that of analytical solutions and R-K-4 Solutions.

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