

Prime Coloring Of Unit Distance And Non Crossing Graphs

P.Murugarajan ,R.Aruldooss

Abstract: In the recent era, the coloring of the graphs is arousing the attention of researchers. Among the mathematical models, coloring technique plays a vital role in a broad range of applications in real life problems and it solves several complexities of computer networks. In this paper, prime coloring and its chromatic number of unit distance and non crossing Graphs are depicted and its results are validated with few theorems **Subject Classification Code:** 05C12, 05C78, 05C15

Keywords: Prime graph, Vertex Coloring, Prime Coloring, Unit distance graph, Non Crossing Graphs

1. INTRODUCTION:

We use the terminology as Douglas B.West [1].let $G=(V, E)$ be neither a Pseudo graph nor a disconnected graph and here order of V be $|V|=n$. In this paper, we discuss Prime graphs and Coloring. The Prime labeling emanate with Entringer *et.al* 1980 [2,3]. Be the graph $G=(V, E)$ with vertex set V is said to have a Prime labeling if there exists a bijection $f: V(G) \rightarrow Z^+ - \{0\}$, Z^+ is the set of positive integers such that for each edge $xy \in E(G)$, $\gcd\{f(x), f(y)\}=1$ [5].It begin to be an interesting problem to investigate the Prime coloring as possible in a graph with n vertices. Moreover, the general graph G is proper colorable but not prime graph forever. The Coloring of a graph is explained as pair wise adjacent vertices receive conflict colors.

2. PRIME COLORING OF UNIT DISTANCE AND NON CROSSING GRAPHS:

In this entire paper, we consider ψ is the bijective function and ψ' is the Chromatic number of G as well as η is also chromatic number of prime graph. The unit distance graph was defined as in on Euclidean plane two vertices connected by on edge with the distance purely one i.e. $d(c_i, c_j)=1$ [7,8] and a graph having no minimum crossing edges is called as non crossing graph i.e. $Cr(G)=0$. Now, Let us define a Prime coloring of graph as follows,

Definition 2.1:

Prime Coloring is defined as G be a loop less and Without multiple edges with n distinct Vertices on Color class $C=\{c_1, c_2, c_3, \dots, c_n\}$ a bijection $\psi: V \rightarrow \{c_1, c_2, c_3, \dots, c_n\}$ if for each edge $e = c_i c_j, i \neq j$, $\gcd\{\psi(c_i), \psi(c_j)\}=1$, $\psi(c_i)$ and $\psi(c_j)$ receive distinct Colors. The Chromatic number of Prime coloring is minimum cardinality taken by all the Prime colors. It is denoted by $\eta(G)$.

Theorem 2.2:

Every path P_n graph is Prime if it is Prime colorable graph.

Proof:

Let the path P_n with n distinct vertices, the bijection

$\psi: V(P_n) \rightarrow \{c_1, c_2, c_3, \dots, c_n\}$ graph if for each edge

$e = c_i c_j$, $\gcd\{\psi(c_i), \psi(c_j)\}=1, i \neq j$,

$\psi(c_i) = c_i$

$\psi(c_j) = c_j$ receive distinct colors. Then the path P_n Graph is Prime colorable graph.

Corollary 2.3: If path P_n graph is prime then $\eta(P_n) = 2$.

Example 2.4: P_n is Prime colorable graph

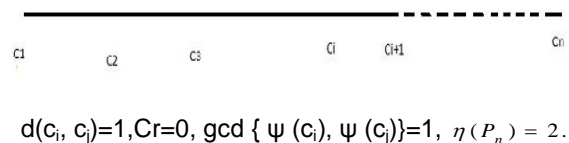


FIGURE 2.1

Theorem 2.5: Every Cycle C_n , $n \geq 4$ if it is Prime colorable graph.

Proof:

Assume that Cycle C_n on $n \geq 4$ vertices with the bijection ψ

$\psi: V(C_n) \rightarrow \{c_1, c_2, c_3, \dots, c_n\}$ graph if for each edge

$e = c_i c_j$, $\gcd\{\psi(c_i), \psi(c_j)\}=1, i \neq j$,

$\psi(c_i) = c_i$

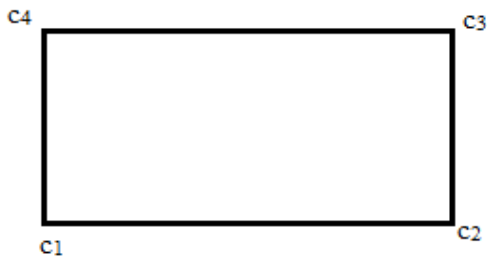
$\psi(c_j) = c_j$, receive distinct colors on cycle C_n on $n \geq 4$.

Corollary 2.6: If Cycle graph C_n , $n \geq 4$ is Prime, then

$$\eta(C_n) = \begin{cases} 2, n \text{ is even} \\ 3, n \text{ is odd} \end{cases}$$

Example 2.7: Cycle C_4

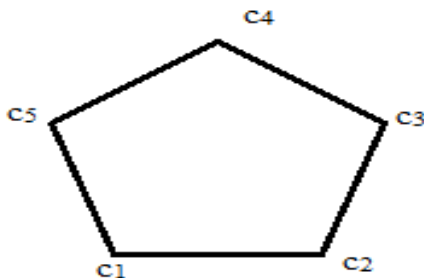
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$d(c_i, c_j)=1, Cr=0, \gcd \{ \psi (c_i), \psi (c_j)\}=1, \eta (C_4) = 2$

FIGURE 2.2

Example2.8: Cycle C_5



$d(c_i, c_j)=1, Cr=0, \gcd \{ \psi (c_i), \psi (c_j)\}=1, \eta (C_5) = 3$

FIGURE 2.3

Theorem2.9: If a Complete graph K_n with $d(c_i, c_j) = 1, Cr = 0, n \leq 3$ is Prime graph, then K_n is Prime colorable graph with $n \leq 3$.

Proof:

Proof by induction $n=1$ obviously holds.

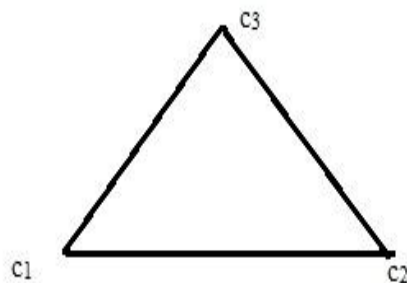
$n=2$ then the graph is a path by the theorem (2.2) holds.

$n=3$ then the graph is triangle every

$\gcd \{ \psi (c_i), \psi (c_j) \} = 1, i \neq j$ then the graph is prime and also receive distinct colors.

$n=4$ then the vertices assigns distinct colors but $\gcd \{ \psi (c_i), \psi (c_j) \} \neq 1$. But $e = c_i c_j$ then the graph is not a prime since $d(c_i, c_j) \neq 1, Cr \neq 0$. Therefore $n > 3, K_n$ is not a prime colorable graph.

Example2.10: The complete graph K_3 is Prime Coloring graph.



$d(c_i, c_j)=1, Cr=0, \gcd \{ \psi (c_i), \psi (c_j)\}=1, \eta (K_3) = 3$

FIGURE 2.4

Example2.11: Complete graph K_4

We observe that K_4-e is unit distance graph with crossing number zero is prime colorable $\eta (K_4 - e) = 3$

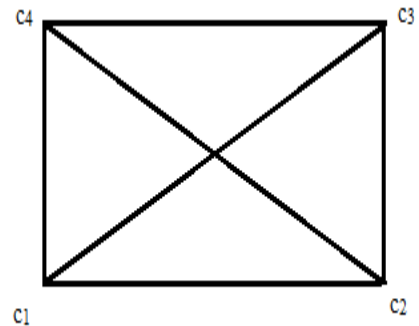


FIGURE 2.5

Here $\gcd \{ \psi (c_1), \psi (c_2) \}=1, \gcd \{ \psi (c_1), \psi (c_4) \}=1, \gcd \{ \psi (c_1), \psi (c_3) \}=1, \text{ But } \gcd \{ \psi (c_2), \psi (c_4) \} \neq 1$.

Consider the general K_4 is proper colorable with

Chromatic number $\psi'(K_4) = 4$,

But $\eta (K_4)$ not defined.

$d(c_i, c_j) \neq 1$, The crossing number of $Cr(K_4) = 0$

EXAMPLE 2.12: Complete graph K_5

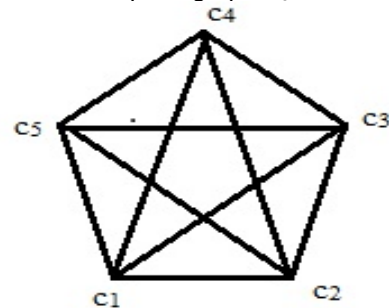


FIGURE 2.6

Here,

$\gcd \{ \psi (c_1), \psi (c_2) \}=1,$

$\gcd \{ \psi (c_1), \psi (c_4) \}=1,$

$\gcd \{ \psi (c_1), \psi (c_3) \}=1,$

$\gcd \{ \psi (c_1), \psi (c_5) \}=1,$

$\gcd \{ \psi (c_3), \psi (c_5) \}=1.$

$d(c_i, c_j)=1$, The crossing number of $Cr(K_5) = 1$

But $\gcd \{ \psi (c_2), \psi (c_4) \} \neq 1$. consider the General K_5 is proper colorable with chromatic number $\psi'(K_5) = 5$.

But $\eta (K_5)$ not defined.

Remark 2.13: Every Complete graph K_n is Proper Colorable but not Prime colorable.

Consider K_n is complete graph pair wise adjacent vertices are connected by the definition of graph coloring it has to assign distinct colors, but the labeled graph K_n with $\{1, 2, \dots, n\}, \gcd(1, n)$ not equal to 1 for $n \geq 3, K_n$ is not Prime graph. Since $d(c_i, c_j) \neq 1, Cr \neq 0$, Thus K_n is not a Prime colorable.

Remark2.14:

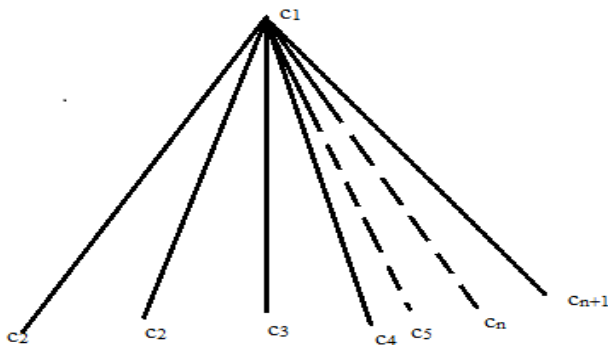
If G is the prime graph with $d(c_i, c_j) = 1, Cr = 0$ and Δ is the maximum degree of the graph then $\eta(G) \leq \Delta + 1$

Theorem2.15: If Star graphs $K_{1, n}$ is Prime graph then $\eta(K_{1, n}) = 2$.

Proof:

Let $V = \{c_1, c_2, c_3, \dots, c_{n+1}\}$ vertices of Star graph $K_{1, n}$, the graph labeled as well as colored, by the bijection $\psi : V(K_{1, n}) \rightarrow \{c_1, c_2, c_3, \dots, c_{n+1}\}$ graph if for each edge $e = c_i c_j, \gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j, \psi(c_i)$ and $\psi(c_j)$ receive distinct colors. Therefore the coloring pairs are at most 2 and also every $\gcd(i, j) = 1$ is a prime graph. Thus, $\psi(c_i) = c_i, \psi(c_j) = c_j$ for $1 \leq i, j \leq n+1$ then every $i \neq j$ is Prime colorable. Therefore minimum coloring needed for the Star graph is $\eta(K_{1, n}) = 2$.

Example 2.16: Star graph $K_{1, n}$



$d(c_i, c_j) = 1, Cr = 0, \gcd\{\psi(c_i), \psi(c_j)\} = 1, \eta(K_{1, n}) = 2$.

FIGURE 2.7

Corollary2.17: Every spanning sub graph of prime graph with $d(c_i, c_j) \neq 1, Cr = 0$ is prime colorable graph. Seoud and youssef [6] proved every spanning sub graph is prime graph. Now coloring proper by the graph become an prime colorable $\gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j$, Since $d(c_i, c_j) \neq 1, Cr = 0, \psi(c_i)$ and $\psi(c_j)$ receive distinct colors. We observe that every spanning sub graph is also prime colorable.

Corollary2.18: Every tree is prime colorable graph.

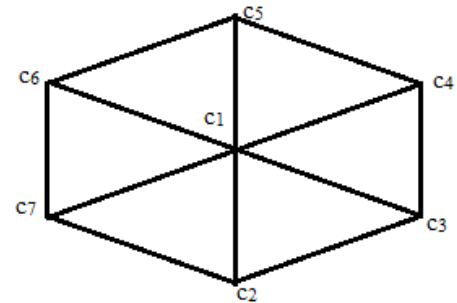
Every tree is prime is proved by Entringer $n \leq 50$ vertices [4]. The tree have been properly colored as the color class $C = \{c_1, c_2, c_3, \dots, c_n\}$. Tree is a spanning sub graph of the general graph G by the above corollary (2.17) holds, there for every tree is prime colorable.

Theorem2.19: If Wheel graph $W_n, n \geq 4$ is Prime graph $d(c_i, c_j) \neq 1, Cr = 0$ then $\eta(W_n) = 4$ when n is even.

Proof:

Let $V = \{c_1, c_2, c_3, \dots, c_{k+1}\}$ vertices of wheel graph W_n . The graph labeled as well as colored. By the bijection $\psi : V(W_n) \rightarrow \{c_1, c_2, c_3, \dots, c_{k+1}\}$ graph if for each edge $e = c_i c_j, \gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j, \psi(c_i)$ and $\psi(c_j)$ receive distinct colors. Therefore the coloring pairs are at most four and also $\gcd(i, j) = 1$ is a prime graph. Thus, $\psi(c_i) = c_i, \psi(c_j) = c_j$ for $1 \leq i, j \leq k+1$ Then every $i \neq j$ is prime colorable. Therefore minimum coloring needed for the Wheel graph is $\eta(W_n) = 4$.

Example 2.20: Wheel graph W_5



$d(c_i, c_j) \neq 1, Cr = 0, \gcd\{\psi(c_i), \psi(c_j)\} = 1, \eta(W_n) = 4$.

we observe that W_7 is only unit distance in Euclid plane with $Cr = 0$ but W_7 is not a prime graph

FIGURE 2.8

Lemma 2.21: G is Prime graph with $d(c_i, c_j) = 1, Cr = 0 \Leftrightarrow G$ is Prime colorable graph.

Proof:

Let us assume G is prime label with the bijection $\psi : V(G) \rightarrow \{c_1, c_2, c_3, \dots, c_n\}$ graph if for each edge $w = c_i c_j, \gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j, \psi(c_i)$ & $\psi(c_j)$ receives distinct colors and $d(c_i, c_j) = 1, Cr = 0$. Now coloring the vertices is proper coloring. Then the resulting graph with different colors. Conversely, G is Prime coloring by the definition of graph coloring holds it is a prime graph with $d(c_i, c_j) = 1, Cr = 0$.

Remark 2.22: If G is Prime Graph with $d(c_i, c_j) = 1, Cr = 0$ then $\psi'(G) = \eta(G)$

Lemma 2.23:

If $\psi'(P_n) = \eta(P_n) = 2$ with $d(c_i, c_j) = 1, Cr = 0$, then the following are equivalent
 (i). P_n is Path.
 (ii). P_n is Prime graph.
 (iii). P_n is Prime colorable graph.

Proof:

(i) \Rightarrow (ii) let the path P_n with n distinct vertices then the chromatic number of the graph $\psi'(P_n) = 2$ we have to prove P_n is a prime graph with

$\gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j$, then P_n is a prime graph.

(ii) \Rightarrow (iii)

By the theorem (2.2) refers to every prime graph P_n which assigned the distinct colors and by the definition of a prime color then $\eta(P_n) = 2$.

(iii) \Rightarrow (i) from the proof given prime colorable graph $\psi'(P_n) = \eta(P_n) = 2, d(c_i, c_j) = 1, Cr = 0$

Then n distinct vertices are connected as a path.

Lemma 2.24:

If $\psi'(C_n) = \eta(C_n) = \begin{cases} 2, & n \text{ is even} \\ 3, & n \text{ is odd} \end{cases}$ with $d(c_i, c_j) = 1, Cr = 0$

then the following are equivalent

(i). C_n is Cycle

(ii). C_n is Prime graph.

(iii). C_n is Prime colorable graph.

Proof:

Case (i) n is odd:

(i) \Rightarrow (ii)

Consider C_n be the cycle with distinct n vertices. If Cycle $C_n, n > 3$ is an odd cycle with $2n-1$ vertices, by the reference [6] Fusion, switching, duplication of the vertices than the graph is the prime graph.

(ii) \Rightarrow (iii)

Consider C_n is prime graph when n is odd, be the fusion, switching, duplication is allowed then the graph is prime by the definition (2.1) C_n is prime colorable graph.

(iii) \Rightarrow (i)

By the definition of (2.1), all the vertices are distinct when n is odd i.e. $c_1 = c_{2n-1}, n > 3$ and holds $\psi'(C_n) = \eta(C_n) = 3$.

Case (ii) n is even

(i) \Rightarrow (ii) Clearly the even number of vertices on the cycle is prime [6] and $\gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j$

(ii) \Rightarrow (iii)

By the theorem (2.5) the graph $\gcd\{\psi(c_i), \psi(c_j)\} = 1, i \neq j$, then $\psi'(C_n) = \eta(C_n) = 2$ pair wise adjacent vertices receive distinct colors.

(iii) \Rightarrow (i)

$\psi'(C_n) = \eta(C_n) = 2$ with $d(c_i, c_j) = 1, Cr = 0$ the Cycle $C_n,$

$n > 3$ we get $c_1 = c_{2n-2}, i \neq j$ vertices are distinct and even vertices of C_n is cycle.

Remark 2.25: Every prime graph is proper colorable graph.

Remark 2.26: Every graph G is proper Colorable but not prime color.

Observation 2.27:

For any graph $G, d(c_i, c_j) = 1, Cr = 0$ is prime graph then G is prime colorable graph.

CONCLUSION:

This article discussed & absorbed the upshot of graph coloring technique which associated with the prime graph. We elucidated the coloring techniques of Unit Distance Graphs and non Crossing Graphs also evinced the some interesting results obtained by Prime Coloring graphs.

CONFLICT OF INTEREST:

The authors confirm that there is no conflict of interest to declare for this publication.

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