The Characteristics Of Partial Lattices And Irreducibility Of Measurable Functions On Partial Lattice

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ABTRACT: By defining the principal meanings of locally-measurable lattice, complete-measure, saturated-lattice measure, we demonstrates that each lattice measure space can be incorporated into a complete-lattice measure space also set up an outcome that If ɲ is lattice sigma-finite measure then it is saturated. Also provides the definitions of join (meet) irreducibility of an element of a partial lattice, measurable function on a partial lattice and proves the join (meet) of two measurable functions defined on partial lattice is measurable, the set of all real-valued measurable functions is a vector space as well as lattice. The main object of this paper is if $\{h_n\}$ is an increasing(decreasing) sequence of join(meet) irreducible measurable functions on a partial measurable lattice space (H, $\overline{\mathcal{B}}$, ɲ) then their join(meet) is join(meet) irreducible measurable function.

Index terms: complete measure, locally measurable lattice, saturated, meet irreducible, join irreducible.

1 INTRODUCTION

[1, 5] Extensively contemplated the ideas of partial lattices on outer measure and their attributes. By [2, 4, 5] the basic meanings of measure, lattice measures, lattice, partial lattice, Boolean lattice and quantifiable characteristics of a partial lattice in a countable Boolean lattice measure are summed up with the assistance of [3]. In 1963, **[6]** introduced a generalization of lattice measure concepts. He explained much about the reducibility and irreducibility of elements of a lattice. If L is a lattice, a ∈ L is said to be join reducible if there exist b, c ∈ L, such that b < a, c < a and b ∨ c = a. If a is not join reducible, a is said to be join irreducible in L. a is said to be meet reducible if there exist b, c ∈ L, such that a < b, a < c and b ∧ c = a. If a is not meet reducible, a is said to be meet irreducible in L. The least element and every atom of a lattice, bounded below is join-irreducible. The greatest element and every dual atom of a lattice bounded above, is meet-irreducible. Every element of a chain is meet-irreducible as well as join-irreducible; it may also happen that every element of a lattice is meet-reducible as well as join-reducible. All through the notations are given by Boolean lattice (L), countable Boolean lattice of partial lattices of L**(L)** and lattice measure (ɲ) characterized on **L.** H be a partial lattice, countable Boolean lattice of partial sublattices of H is ($\overline{\mathcal{B}}$) and measure on $\overline{\mathcal{B}}$ is (ɲ). Section2, provides meaning of locally-measurable lattice, complete-measure, saturated-lattice measure, and gives an outcome that In **L,** if $\{N_k\}$ be any countable collection null partial lattices like $N_k \subset N_{k+1}$ then so their join is a null partial lattice in **L**. Section3, demonstrate that each lattice measure space can be incorporated into a complete-lattice measure space, and set up an outcome that If ɲ is lattice sigma-finite measure then it is saturated. Section4, provides join (meet) irreducibility of an element of a partial lattice H, measurable function on a partial lattice, join (meet) of two functions on a partial lattice also proved the equivalent conditions, algebraic conditions of measurable functions defied on H. Section5, provides theorems viz the join (meet) of two measurable functions defined on partial lattice is measurable, the set of all real-valued measurable functions is a vector space as well as lattice also If $\{h_n\}$ is an increasing (decreasing) sequence of join (meet) irreducible measurable functions on a partial measurable lattice space (H, $\overline{\mathcal{B}}$, ɲ) then their join(meet) is join(meet) irreducible measurable function.

2 Lattice measure space (L, L, ɲ)

Let (L, L, ɲ) be a lattice measure space In L, ɲ is said to be complete-measure if all partial lattices of sets of measure zero (i.e if N ∈ L, ɲ(N) = 0 and M ⊆ N implies M ∈ L) H ∈ L is said to be a locally-measurable lattice, if H ∧ N ∈ L for every N ∈ L with ɲ(N) < ∞. If every locally-measurable partial lattice is lattice measurable, then ɲ is called saturated.Observation2.1 All locally-measurable partial lattices form a Sigma-algebra containing L. Result2.1 In L, if $\{N_k\}$ be any countable collection null partial lattices such that $N_k \subset N_{k+1}$ then so their join is a null partial lattice in L. Proof Evidence let N_1, N_2,…… be a succession of null partial lattices. i.e., for every positive whole number n, $N_n \in$ L such that $M_n \subseteq N_n$ Furthermore $ɲ(N_n) = 0$. Unmistakably $\bigvee_{n=1}^{\infty} M_n \subseteq \bigvee_{n=1}^{\infty} N_n$ Presently $\bigvee_{n=1}^{\infty} N_n \in$ L [clearly each $N_n \in$ L by the meaning of L] Presently ɲ($\bigvee_{n=1}^{\infty} N_n$) ≤ $\sum_{i=1}^{\infty}$ $ɲ(N_n)$= 0 [since $ɲ(N_n) = 0$ ∀ n] Along these lines the biggest partial lattice $\bigvee_{n=1}^{\infty} N_n$ whose lattice measure is zero Thus $\bigvee_{n=1}^{\infty} N_n$ is likewise a null partial lattice.

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3 COMPLETE- MEASURE SPACE (L, **L,** ɲ)

Theorem3.1 For (L, L, ɲ) be lattice measure space, we locate (L, L_0, $ɲ_0$) a complete-measure space such that L ⊆ L_0, if P∈ L then ɲ(P) = $ɲ_0$(P), and also P∈ L if and only if P = M ∨ N, where N ∈ L and M ⊆ T, T ∈ L, ɲ(T) = 0. Proof Compose L_0 = {P ⊆ L / P = M ∨ N, N ∈ L, M ⊆ T, T ∈ L, ɲ(T) = 0} = {M ∨ N / N ∈ L, M ⊆ T, T ∈ L, ɲ(T) = 0} Clearly ϕ ≠ L ⊆ L_0 [since ϕ = ϕ ∨ ϕ, ϕ ⊆ ϕ, ϕ ∈ L, ɲ(T) = 0] To prove L_0 is a sigma-algebra, let P ∈ L_0 implies P = M ∨ N, N ∈ L, M ⊆ T, T ∈ L, ɲ(T) = 0. Now P′ = P′ ∧ L = P′ ∧ (T ∨ T′) = (M′ ∧ N′) ∧ (T ∨ T′) [since P = M ∨ N] = (M′ ∧ N′∧ T) ∨ (M′ ∧ N′ ∧ T′) = [(M ∨ N)′∧ T] ∨ (N′ ∧ T′) [since C′ ⊆ A′] = [(M ∨ N)′∧ T] ∨ (N ∨ T)′ = (N ∨ T)′ ∨ [(M∨ N)′∧ T] = (N ∨ T)′ ∨ [T – (M ∨ N)] Here (N ∨ T)′ ∈ **L**, T – (M ∨ N) ⊆ T and ɲ(T) = 0. It means P′ ∈ $\mathbf{L}_0$. Also let {P_n} be a sequence of partial lattices from $\mathbf{L}_0$. Then P_n = M_n ∨ N_n, where N_n ∈ **L**, M_n ⊆ T_n, T_n ∈ **L**, ɲ(T_n) = 0. Now $\vee_n P_n = \vee_n (M_n \vee N_n)$ = $[(\vee_n M_n)] \vee [(\vee_n N_n)]$, $\vee_n N_n$ ∈ **L**, $\vee_n M_n \subseteq \vee_n T_n$ ∈ **L**. Also ɲ($\vee_n T_n$) ≤ $\sum_n$ ɲ(T_n) = 0 Implies ɲ($\vee_n T_n$) = 0. Therefore $\vee_n P_n$ ∈ **L** Hence $\mathbf{L}_0$ is sigma-algebra Define $ɲ_0$ on $\mathbf{L}_0$ by $ɲ_0$ (P = M ∨ N) = ɲ(N), ∀ P∈ $\mathbf{L}_0$. Obviously $ɲ_0$ is well defined. If P = M_1 ∨ N_1 and P = M_2 ∨ N_2, where N_1, N_2 ∈ **L**, M_1 ⊆ T_1 and M_2 ⊆ T_2, ɲ(T_1) = 0, ɲ(T_2) = 0 then ɲ(N_1) = ɲ(N_2) For ɲ(N_1) ≤ ɲ(M_1 ∨ N_1) = ɲ(M_2 ∨ N_2) ≤ ɲ(T_2 ∨ N_2) ≤ ɲ(T_2) + ɲ(N_2) = ɲ(N_2) Therefore ɲ(N_1) ≤ ɲ(N_2) also similarly ɲ(N_2) ≤ ɲ(N_1) Since ɲ is non-negative, we have $ɲ_0$ is non-negative $ɲ_0$(ϕ) = $ɲ_0$(ϕ ∨ ϕ) = ɲ(ϕ) = 0 Again let {P_n} be a sequence of pairwise disjoint sets in $\mathbf{L}_0$ Implies $ɲ_0(\vee_{n=1}^{\infty} P_n) = ɲ_0 [\vee_{n=1}^{\infty} (M_n \vee N_n)] = ɲ_0 [(\vee_{n=1}^{\infty} M_n)] \vee [(\vee_{n=1}^{\infty} N_n)] = ɲ(\vee_{n=1}^{\infty} N_n) = \sum_{n=1}^{\infty} ɲ(N_n) = \sum_{n=1}^{\infty} ɲ_0(N_n)$ So (L, $\mathbf{L}_0$, $ɲ_0$) is a lattice measure space. To show complete measure space, let P ∈ $\mathbf{L}_0$ with $ɲ_0$(P) = 0 Since P ∈ $\mathbf{L}_0$, we have P = M ∨ N, N ∈ **L**, M ⊆ T, T ∈ **L**, ɲ(T) = 0. Now 0 = $ɲ_0$(P) = $ɲ_0$(M ∨ N) = ɲ(N) Let F ⊆ P Now F = P ∧ F = (M ∨ N) ∧ F = (M ∧ F) ∨ (N ∧ F) ⊆ M ∨ N ⊆ T∨ N Therefore F = F ∨ ϕ, where ϕ ∈ **L**, F ⊆ T ∨ N, T ∨ N ∈ **L**, ɲ(T ∨ N) = 0. Implies F ∈ $\mathbf{L}_0$ Hence the theorem Result3.1 If ɲ is lattice sigma-finite measure then it is saturated. Proof Assume ɲ is a lattice sigma-finite measure. Then L = $\vee_n Y_n$, ɲ(Y_n) < ∞, ∀ n. Suppose Y_n's are disjoint (Without loss of any generality) Let K be a locally-measurable partial lattice, then K ∧ Y_n ∈ **L** and ɲ(K ∧ Y_n) ≤ ɲ(Y_n) < ∞, ∀ n. Along these lines K = K ∧ Y = K ∧ ($\vee_n Y_n$) = $\vee_n$ (K ∧ Y_n), ∀ n Which implies K ∈ **L** [since K ∧ Y_n ∈ **L**, ∀ n and L is a sigma-algebra] Hence ɲ is saturated.

4 IRRDUCIBILITY OF AN ELEMENT IN A PARTIAL LATTICE

Definition4.1 an element a in H is said to be join reducible in H if there exist c, b, c ≠ a, b ≠ a, b ∨ c exist in H and b ∨ c = a. If a is not join reducible in H, a is said to be join irreducible in H. a is said to be meet reducible in H if there exist c, b, c ≠ a, b ≠ a, b ∧ c exist in H and b ∧ c = a. If a is not meet reducible in H, a is said to be meet irreducible in H. Example4.1 The set of natural numbers is a lattice with a join b = l.c.m of {a, b} and a meet b = g.c.d of {a, b}. If n is prime, for n ≥ 1, then this lattice is irreducible. Example4.2 In the partial lattice H of positive integers greater than 1 that are not prime nor for some n of the form n(n+1) in which a join b = l.c.m of {a, b} and a meet b = g.c.d of {a, b}. 4, 8, 9, 10, 14, 15 are join irreducible where as 36 = 4 ∨ 9 is not. Definition4.2 (measurable function on a partial lattice) If the set {l∈ H: h(l) > a } = h^{-1}(a, ∞) is measurable for all real numbers a, then we say that h is a real valued measurable function defined on a partial lattice H. Theorem4.1 If h: H → R^{∞}, then every a ∈ □, the sets {l∈ H : h(l) ≥ a}, {l∈ H : h(l) < a}, {l∈ H : h(l) ≤ a}, { l ∈ H : h(l) > a} are equivalent. Proof By [3] the above statement is equivalent. Definition4.3 The extended real valued functions f and g are defined on partial lattice H then join, meet are defined by f ∨ g = max(f(l), g(l)) and f ∧ g = min(f(l), g(l)) for any l ∈ H. Theorem4.2 [3] f: H → □, g: H → □ are measurable then so are (1) f ± g (2) cf (c ∈ □) (3) f^2 (4) fg Proof By [3] the above statement is measurable.

5 IRREDUCIBILITY OF MEASURABLE FUNCTIONS ON A PARTIAL LATTICE

Let H be a partial lattice, $\overline{\mathcal{B}}$ be a countable Boolean lattice of partial sublattices of H and η be a measure on $\overline{\mathcal{B}}$. Theorem5.1 If f: H → □, g: H → □ are measurable then so is f ∨ g, f ∧ Proof $(f \vee g)^{-1}$ (-∞, a) = {l∈ H: max (f(l), g(l)) < a } = {l∈ H: f(l) < a and g(l) < a = {l∈ H: f(l) < a } ∧ {l∈ H: g(l) < a } = f^{-1}(-∞, a) ∧ g^{-1}(-∞, a) $(f \wedge g)^{-1}$ (a, ∞) = {l∈ H: min (f(l), g(l)) > α} = {l∈ H: f(l) > α and g(l) > α} = {l∈ H: f(l) > α} ∧ {x / g(l) > α} = f^{-1}(a, ∞) ∧ g^{-1}(a, ∞) Theorem5.2 The set η(H) of all real-valued measurable functions is a vector space as well as lattice. Proof That η(H) is a vector space over □ follows from the fact that the zero function belongs to η(H) and that η(H) is a lattice follows from Theorem5.1 Observation5.1 If g and h are measurable and incomparable, g ∨ h as well as g ∧ h are measurable. If further g and h are both join (meet) irreducible, g ∨ h (g ∧ h) is clearly not join (meet) irreducible. However in the case of infinite sequences, we have the following theorems 5.3. and 5.4. Theorem5.3 If {h_n} is an increasing sequence of join irreducible measurable functions on a partial measurable lattice space (H, $\overline{\mathcal{B}}$, ɲ) then their join is join irreducible measurable function. Proof <u>Measurability:</u> write h = $\vee_{n=1}^{\infty} h_n$ For all a ∈ □, h^{-1}([a, ∞)) ⇔ h(l) ≥ a ⇔ h_n(l) ≥ a for some n ⇔ l ∈ h^{-1}([a, ∞)) ⇔ l ∈ $\vee_{n=1}^{\infty} h_n^{-1}$([a, ∞)) Hence follows measurability of h <u>Irreducibility:</u> To prove join irreducibility of h, assume that h = g ∨ m where g, m are measurable functions on H. For any positive integer k, h_k= h_k ∧ h = h_k ∧ (g ∨ m) = (h_k ∧ g) ∨ (h_k ∧ m). Since h_k is join irreducible and h_k ∧ g and h_k ∧ m are measurable either h_k = h_k ∧ g or h_k = h_k ∧ m is for every k, either h_k ≤ g or h_k ≤ m. Let A = {k / h_k ≤ g}, B = {k / h_k ≤ m}. Since every k must be either in A or B, one of A, B must be infinite. If A is infinite then for every n there exists k_n ∈ A such that k_n > n. k_n > n ⇒ h_{k_n} ≥ h_n k_n ∈

A ⇒ g ≥ h_{k_n} Hence g ≥ h_{k_n} ≥ h_n for all n Hence g ≥ $\bigvee_{n=1}^{\infty} h_n$ = h g ∨ m = h ⇒ g ≤ h Hence f = g.If B is infinite we get, by a similar argument that h = m. h = g ∨ m ⇒ f = g or h = m Hence h is join irreducible. Theorem5.4 If {h_n} is decreasing sequence of measurable meet irreducible functions on a partial measurable lattice space (H, $\overline{\mathcal{B}}$, ɲ) then h = $\bigwedge_{n=1}^{\infty} h_n$ measurable and irreducible. Proof <u>Measurability:</u> Let a ∈ □. If h(l) ≥ a then h_n(l) ≥ a for all n. Implies {l ∈ H: f(l) ≥ a} ⊆ $\bigwedge_{n=1}^{\infty}$ {l ∈ H: h_n(l) ≥ a} ----------------- (1) If h_n(l) ≥ a for all n then h(l) = $\lim_{n} h_n(l)$ ≥ a. Hence $\bigwedge_{n=1}^{\infty}$ {l ∈ H: h_n(l) ≥ a} ⊆ {l ∈ H: h(l) ≥ a} ----------------- (2) From (1) and (2) {l ∈ H: h(l) ≥ a} = $\bigwedge_{n=1}^{\infty}$ {l ∈ H: h_n(l) ≥ a} Since h_n is measurable for all n, {l ∈ H: h_n(l) ≥ a} is measurable, hence {l ∈ H: h(l) ≥ a} is measurable, hence h is measurable. <u>Irreducibility:</u> Suppose h = g ∧ m. Then for all n, h_n = h_n ∨ h = h_n ∨ (g ∧ m) = (h_n ∨ g) ∧ (h_n ∨ m). Since h_n is meet irreducible, either h_n = h_n ∨ g or h_n = h_n ∨ m That is; either h_n ≥ g or h_n ≥ m. Let A = {n ∈ H: h_n ≥ g}, B = {n ∈ H; h_n ≥ m}. As in theorem5.3 either A or B must be infinite. If A is infinite then for every n there exists k_n > n in A so that h_n ≥ h_{k_n} ≥ g. Since h_n ≥ g for all n, h = $\bigwedge_{n=1}^{\infty} h_n$ ≥ g. If B is infinite, a similar argument yields h ≥ m. Thus either h ≥ g or h ≥ m Since h = g ∧ m, h ≤ g and h ≤ m. Hence h = g or h = m Hence h is meet irreducible.

CONCLUSION

We proved that each lattice measure space can be incorporated into a complete-lattice measure space, and established a result that If ɲ is lattice sigma-finite measure then it is saturated also provided the definitions of measurable function on a partial lattice and verified the equivalent conditions of measurable functions, the measurability of sum (difference) functions, constant multiplication of function, square of function, product of functions, join (meet) of two functions, the set ɳ(H) of all real valued measurable functions is a vector as well as lattice, finally proved that if {h_n} is an increasing (decreasing) sequence of join (meet) irreducible measurable functions on partial lattice (H, $\overline{\mathcal{B}}$, ɲ) then their join (meet) is join (meet) irreducible measurable function.

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