# What Error Happened To Inferences Of Senior High School Students Using Mathematical Critical Thinking Ability?

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**Abstract:** Critical thinking is one of important factors in learning mathematics because it is used as one of the tools to solve mathematical problems. In the process of mathematical critical thinking, students experience the process of making inferences i.e. in the process of making conclusions of the premises given so that the premises and the conclusions form valid arguments. However, students sometimes make mistakes in giving reasons to the inferences made. The objective of this research was to analyze the error of a high school student in making inferences using mathematical critical thinking ability based on APOS theory. The type of research used was descriptive with qualitative approach. The subjectof this research was a student of twelfth grade. The methods used in the study included two test of essays, interview, and triangulation methods. The findings showed that the student was able to make inferences of given problems. However, the student had not been able to give the reasons of the inferences made. The causes of student errors in working on the questions of inferences based on APOS theory were: (1) errors in manipulating the application of the rules of making conclusion on logical thinking, (2) misconceptions (SPLDV) about the equation of coinciding lines, (3) errors dealing with kinds of numbers in mathematics (4) difficulties in changing given premises in the forms of mathematical symbols (5) difficulties in remembering the wayof eliminating and substituting of SPLDV principles, (6) difficultiesin changing SPLDV matrix forms into SPLDV algebra.

**Keywords**: Error Analysis, Inference, Mathematical Critical Thinking Ability, APOS Theory.

## 1 INTRODUCTION

Critical thinking is one of the tools used in everyday life to solve problems. A person is indirectly confronted with a decision-making process involving an interpretation, explanation, analysis, evaluation, and conclusion in everyday life. The idea is supported by [19], who say that critical thinking skills involve the ability to analyze, interpret, evaluate, summarize, and synthesize all information. Further, [7] says, critical thinking has become one of the tools used in everyday life to solve some problems as it involves logical reasoning, interpreting, analyzing and evaluating information to make a correct and valid decision. However, students are sometimes difficult to develop critical thinking skills due to the difficulties of students in understanding a problem, especially on mathematics subjects. It happens because almost all the teaching learning processes of mathematics do not familiarize students to think critically. This is contrary to idea of [12] which says that the curriculum of mathematics education worldwide should seek the development of critical thinking skills. These important critical thinking skills need to be developed in everyday life, especially in learning mathematics because the problems of mathematics require a complex concept. One activity that can develop students' critical thinking ability is by giving mathematical problems dealing with mathematical critical criticism indicators. As [10] says that students can develop their ability in mathematical critical thinking when they face math problems; identify possible solutions; evaluate; and justify the reasons for the results so these acticities enable students to become confident critical thinkers. In addition, the development of critical thinking can be done by creating conducive classroom atmosphere. [15] say, students are expected to use mathematics and mathematical mindset in everyday life to study various types of science that emphasize the logical arrangement and character building of students as well as the ability to apply mathematics. These important critical thinking skills set experts to make indicators of critical thinking ability. According to [8], there are six basic components in critical thinking skills: focus, reason, inference, situation, clarity, and overiew. Further, according to [9], there are six skills in critical thinking skills namely interpretation, analysis, evaluation, inference, explanation, and self regulation. Then according to [4], there are nine indicators of critical thinking ability: Claim/Argument Identification, Premises and Conclusion Identification, Possible Assumption(s) Identification, Inferencing, Validating, Other points of view Identification, Comparing & Contrasting of Claims/Conclusions, Decision Making, Communicating Ideas. From some of these indicators there is an Inference Indicator which is very important in critical thinking skills. Inference is the ability that can identify and get the elements needed in drawing conclusions. In this case, [7] says that in making inference, students must identify and acquire reasonable elements, make guesses and hypotheses, and summarize the consequences of the data. Therefore, when a student is able to make inferences, then he can conclude a mathematical problem he is facing. In addition, a studentthat is able to make inferences, then he can conclude more than one of the premises given. Therefore, by having the ability to make inferences, a student is expected to be able to make decisions on the issues faced. Indicators of making inference ability in critical thinking are sometimes forgotten by some students. The proof shows that some students are able to make inferences, but they are unable to give the reason. The predicted cause of this inability is an error in the mental process of understanding the concepts of mathematics they are working. Therefore, students' mental process in making inferences must be controlled. The APOS theory is used to check students' mental process. According to [3], the APOS theory generally consists of four stages; they are: actions, processes, objects, and schemes. Action is the ability to understand concepts by using external stimuli. Process is the ability to think of the same action based on specific input and output without having to do explicitly. Object realizes a process as a whole--realizing,and transforming the action based on its entirety as well as building explicit transformation.Scheme is the ability to build relationships of actions, processes, objects, and other schemes to solve problems. By using the stages of APOS theory, it is expected that the errors occur when they are working on maths problems are known. There have been many studies discussing students' critical thinking ability in making inferences. [1], concluded of the five indicators of critical thinking ability measured, the inference indicator got the lowest result because the students were not good at distinguishing level of the right or wrong conclusions drawn from the data provided. According to [16] prospective teachers had high indicators of critical thinking skills in induction and observation while the indicators of deduction and assumption identification were low. [14] concluded that the components of mathematical reasoning and interpretation contributed 75% and 55% to critical thinking skills respectively. The components of analysis, evaluation, and inference contributed 60%, 55%, and 48% respectively. These results indicate that the inference component produces the lowest percentage compared to the other components of critical thinking ability. [11], concluded that the magnitude of critical thinking skills in terms of gender and college majors varies based on the Watson-Glaser model of critical thinking ability consisting of five variables: inference, assumption, deduction, interpretation and Argument. Previous researches proved that the inference indicator of critical thinking ability produced the lowest percentage compared with other indicators. However, the results of the studies showed that no one has examined why the ability of inference get the lowest percentage. Therefore, the purpose of this study was to analyze what errors occur when a student of senior high school (SMA) made inferencesby employing his critical thinking ability in solving math problems using APOS theory.

## 2 METHOD

The research was a descriptive research with qualitative approach. In this research, one student of twelfth grade was selected as the research subject. The reason for taking this subject was because the twelfth grade students had learned about the material problem used in this research. The methods used included test, interview, and triangulation methods. The main instrument in this research was the researchers themselves equipped with test and interview instruments. The tests were inference tests in mathematical critical thinking skills consisting of two essay questions. The student worked on the questions that had been made. Then the test results were confirmed through interviews with the selected subject. The interviewwas semi-structured interview because the interview guidelines had been prepared first based on student answers, but the questionswere developed in accordance with the circumstances and characteristics of the research subject. The interview was flexible and allowed the researchers to follow the subject's thinking without switching from the initial goal of the interview to find out the student's mistake in making the inference of mathematical questions given using APOS theory. Then, using data triangulation method,the researchers compared the results of mathematical test questions with the results of the interview. The following were questions of inference tests in mathematical critical thinking skills used in this research:

1. The following statements are the premises of an argument. To be a valid argument, write a statement that concludes from one or all of these premises:

   **Premise 1**: if $A$ is an order 2 × 2 matrix, then the determinant value of the matrix $A$ is $|A|$ = ad - bc

   **Premise 2**: if $A$ is an order 2 × 2 matrix, then the inverse matrix of $A$ is $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

   **Premise 3**: The determinant value of matrix $A$ is not ad-bc

   **Premise 4**: Matrix $A$ has no inverse

   **Premise 5**: Matrix $A$ is not an order 2 × 2 matrix

   **Premise 6**: if the inverse matrix $A$ is $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, then the transpose matrix $A$ is $A^{t} = \frac{1}{ad-}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

   What conclusions can you take to obtain a valid argument? Explain the reason in your answer!

2. Notice the following premises:

   **Premise1**: if$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, then the value (x, y) that meets is$\{1,1\}$

   **Premise 2**: if$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, then the value (x, y) that meets is $\{\frac{1}{2}, \frac{3}{2}\}$

   What conclusions can be drawn from these premises! Explain the reason in your answer!

## 3 RESULT AND DISCUSSION

The results of inference questions on mathematical critical thinking skills got from a student. Below were the results of written answers from him: In question number 1, he got four conclusions from the premises given in the questions so that it formed an argument i.e.from premise (1) and (3) he concluded that the matrix $A$ was not a 2 × 2 order; Premise (4) and (5) he concluded that only a 2 × 2 matrixorder which had inverse; Premise (2) and (5) he concluded that the inverse of matrix $A$ was not $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$; Premise (1) and (5) he concluded that the determinant of matrix $A$ was not $|A|$ = ad-bc. In question number 2, he concluded that the value (x, y) had many sets of solutions. To obtain information from the student about the problems, it was necessary to conduct an interview. The purpose of the interview was to confirm the answers that he had been working on. Then, to identify his errors in making inferences on math problems given used the APOS theory. Below were the results of the interview: In question number 1: he was asked where the conclusions obtained, then heans wered the conclusions were obtained from the merger of two premises and based on the logic, a valid conclusion was obtained. In question number 1: he was asked whether there were other conclusions from the premises, then he answered that there were no other conclusions because there were only four conclusions got from the premises given. In question number 1: he was asked what a premise was, he replied a premise was a statement of an argument and their combination formed a causal relationship. In question number 1: he was asked what the conclusion was, he answered conclusion was one's statement about the combination of the premises given. In question number 1: he was asked about premise (1) and (2) whether the argument was valid or not, he answered it was not a valid argument because a conclusion could not be made from it. He also answered that premise (1) and (4) was not a valid argument because it did not make sense; premise (1) talked about determinant, while premise (4) discussed about inverse. Premise (3) and (4) also did not form a valid argument because there was no conclusion. In question number 2: he was asked why he answered there were many sets of solutions. He answered from premises (1) and (2) it was clear that the set solutions of (x, y) was (1,1) and $\{\frac{1}{2},\frac{3}{2}\}$. Since the set of completions was more than one, he concluded that there were many sets of solution. In question number 2: he was asked if there was another set of solution, then he answered there was another set (0.75, 1.25). The reason was the result of the substitution was 2. Further, he said it was the last set and no other possibilities. In question number 2: he was asked what numbers (1,1) were. He answered they were whole numbers. Further question was what whole number was.He answered whole number wasa number that had no remains. In question number 2: he was asked what numbers $\left\{\frac{1}{2},\frac{3}{2}\right\}$ were. He answered they were natural numbers. The next question was what a natural number was. He answered it was a number in the form of fraction. Then, what number (0.75; 1.25) were, he answered the same as$\left\{\frac{1}{2},\frac{3}{2}\right\}$, they were natural numbers. However, they were different in the forms. In question number 2: he was asked how and where he got $(1,1)$, $(\frac{1}{2},\frac{3}{2})$, and $(0.75; 1.25)$. Then he answered principally two values of $(x, y)$ had to be found and if they were added the result was 2. In short, there were a lot of sets of solutions. The answers and the interview showed that errors made by the student in answering inference questions were as follows: The analysis for question number one, he could not apply the rules of drawing a logic conclusion in giving reason of the inferences made of the premises given. It was proven from the results of the interview. In making inferences, he built them only upon his minds without associating them with the concepts of mathematical logic. Then for question number two, he could not use the concept of a two-variable linear equation system (SPLDV) about two coinciding linear equations that have many solutions from the SPLDV problemsin the form of matrix. The evident could be seen from the interview results, he got another set of solution by trying the values of x and y so that if x and y were added the result was 2. In addition, he did not know about the concept of various kinds of numbers as the bases of the existance of solution sets of a SPLD in matrix form. The results of the interview showed that he made mistakes in giving the definition of natural number, whole numbers, and integers. He also answered incorrectly when he was asked to mention examples of natural numbers, whole numbers, and integers. Based on questions number one and two, at the stage of action in the APOS analysis he had limited understanding about the rules of making logical conclusion and the concepts of coinciding SPLDV. At the previous school level, he only learned about making conclusions that were not related to the rules of drawing logical conclusions and doing SPLDV which had only one solution. The errors in making conclusion of SPLDV and coinciding concepts based on APOS analysis mostly occured in prior learning which was concerned more on memorization and procedures beyondhis understanding. This is in line with the results of research [17] which state that students' mistakes on trigonometric derivative functions based on APOS theory analysis are mostly derived from previous mathematical learning and more than generalizations of certain mathematical rules. His previous learning was dominated by rote learning routines or procedures without understanding them. Another analysis, the studentwas not able to change the given premises into the form of mathematical symbols. It was shown from the results of the interview about question number one. He could not answer when he was asked to mention the formula of ponnen mode, tollen mode, and syllogism in decision-making the rules. In question number two, he was unable to remember the way of elimination and substitution of SPLDV. It was shown from the results of the interview which showed that he could not answer when he was asked to mention the process of substitution and elimination, even though he had tried to answer the problem by entering the values of x and y to the equation and if the valueswere added the result was 2. It indicated that he was not a ware if what he did was the process of substitution. In addition, in question number two he was not able to change the form of SPLDV matrix into the form of SPLDV algebra. The interview implied that he did not think that question number two was a matter of SPLDV algebra that was converted into matrix multiplication. The findings showed that hewas not successful in process and object stages of APOS theory because he could not perform the problem-solving steps well or change into mathematical symbols. In short, he could not make the inference properly. Therefore, if he wanted to succeed in process and object stages, he did not only use the symbols in math but also have to make conclusions along with the reasons. It is reinforced by [6] who state that students not only use notations or symbols but also must be able to provide mathematical reasons, make conclusions, and generalizations in the way of students themselves. Further analysis indicated that the student could not access the appropriate rules to be applied so that he was wrong in interpreting a problem. The interview on question number one, he made four inferences, but he could not mention the process of making the inferences. It was seen from his inability to provide inferences when hewas given different premises. In question number two, he could name another set of SPLDV in the form of matrix, but he could not explain the way it worked. During the interview, he simply replied that the set could be (x, y) = (0.75, 1.25) because when the two were added the result was 2 so it matched with the equation asked in the question. Based on these findings, he was unsuccessful at the scheme stage of the APOS theory because he was able to answer the questions, but unable to provide a good reason for the answers expressed so that he could not relate between the concepts used in the matter in a scheme. This happens because he did not have a good understanding of mathematical concepts. This is in line with the results of [5], students could answer derivative questions, but they did not have a good understanding of mathematical concepts so they make conceptual mistakes. Based on the explanation, in general, the student’s responses in solving the problems of inference on the ability of mathematical critical thinking was at the level of action on the theory of APOS. This is reflected in two questions that had been given, he could complete two questions, but he could not mention the process through which to answer the two questions. He could not make a concept into a process. From the results, it appears that the scheme of concepts of logical decision-making and the concepts of coinciding SPLDV was not assimilated into his cognitive structure. It made him fail to coordinate a scheme with other schemes. The study revealed that some aspects of APOS theory were not fully operational, because he was not successful in recalling the concepts used in solving the given problem. He could not manipulate the concept well so that he could not follow the instructions given in applying the rules that had been memorized. Therefore, teachers need to be aware that students should learn conflict so thatthey can reinforce new concepts that they are facing. Errors identification based on APOS theory that students do in the inference questions on critical thinking skills helps teachers to know the weaknesses of students and where the weaknessescome from. From the mistakes that have been found, then teachers must use learning tools to improve students' ability so that it give impact on the development of students' critical thinking skills of math. According to [13], several ways that can improve the ability to think math critically both middle school and post-secondary level, among others: using active learning strategies involving students in the learning process, the focus of learning on the learning process is not on the results, and use assessment techniques which give the students an intellectual challenge rather than an assessment that is concerned with memory or rote. Furthermore, according to [18], the effectiveness of critical thinking teaching is influenced by the conditions in the learning environment consisting of teaching variables (learning strategies and critical thinking teaching approaches), and the variables related to students (the level of the year and the academic achievement of the students). Therefore, with the development of students' critical thinking skills in mathematics, students will be able to have a mental structure based on APOS theory. This is in line with [2], study which said students' improvement on mathematical critical thinking skills would enable students to look at everything from various aspects and provide an opportunity to find formulas and rules.

## 4 CONCLUSIONS

Based on data analysis and discussion of the research, it can be concluded that the student was able to make conclusions from the questions of inference in the ability of mathematical critical thinking. However,he had not been able to give the reason of the inferences made. This was caused by mistakes in the mental structure that he experienced. The student's errors based on the theoretical framework of APOS were (1) the error in manipulating the application of rules of logic when giving the reason of the inference made based on the premises given, (2) the error in using the concept (SPLDV) about two coinciding linear equations that have many solutions to the SPLDV problem in matrix form, (3) the errors about the concepts of kinds of numbers as the bases to determine the presence or absence of the solution sets of a matrix SPLDV, (4) the student was incapable of changing the given premises into the form of mathematical symbols, (5) the student did not remember the way of elimination and substitution on the SPLD, (6) the studentwas unable to change the SPLDV matrix form into SPLDV algebra. The findings of this study provide an illustration that teachers as educators should use an appropriate model of learning so that the errors on structural mental of students based on APOS theory in working on the question of inference on the ability of critical thinking will not happen again. Therefore, it is expected that further research on developing mathematics learning model that can improve students' ability in making inference in critical thinking is conducted.

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